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Model theoretic algebra, by Christian U. Jensen and Helmut Lenzing. Gordon & Breach, New York, 1989, 430 pp., $59.00. ISBN 2-88124-717-2

What is model-theoretic algebra? According to [4], it is the subject which "deals with algebraic structures or theories, aiming at model-theoretic results or using model-theoretic means." This definition does not strictly cover some important early papers where model-theoretic results and concepts were presented... without using model-theoretic means. The papers of Artin and

Schreier introducing the theory of real-closed fields and offering a retrospectively clear vision of "real algebra" (to use their own words) are omitted in [4], while Hermann's famous 1926 paper is included: Hermann's paper was the first "modern" one discussing constructive commutative algebra; it gives inter alia an algorithm to settle "does P belong to the ideal generated by $Q_1, \ldots, Q_n$?" where P and the Q_i range over polynomials with coefficients in a field. While Hermann's methods had nothing to do with model theory, the subject was later on investigated by model-theoretic methods, notably by Abraham Robinson and van den Dries. Although this is a bit far from our subject, it should be mentioned that other avenues are open and that Kreisel emphasized the use of proof-theoretic methods to extract effective bounds from classical results. Constructive algebra which has numerous connections with model theory, especially when Hilbert's Nullstellensatz or sums of squares are involved, is now pursued per se.

We will not try to delimit more precisely the subject but everyone realizes that there has been a quantitative explosion. [4] records twenty papers between 1926 and 1950 and about one hundered twenty for 1984. The most important of the "first twenty" were those of Tarski [20, 21] (elimination of quantifiers and decidability for the field of real numbers and for the field of complex numbers), Hewitt [8] (an anticipation of nonstandard analysis and of the ultraproducts construction), B. H. Neumann [13] and Higman, B. H. Neumann and H. Neumann [9] (the beginning of the theory of existentially closed groups), Szmielew [18] (model-theoretic study from scratch of the theory of abelian groups; there have been numerous generalizations to modules, more algebraic and less painful), the one of Hermann [7] previously mentioned and a seminal run of papers by Maltsev, in particular [11] which clearly shows that in 1941 Maltsev mastered the method of diagrams and had various applications to group theory of greater technical virtuosity than the possibly more basic applications to algebra later on found by the rediscoverers of the method, Henkin [6] and Abraham Robinson (numerous papers, see [16]). It is interesting to note that in 1940 Maltsev [10] had already obtained fundamental results which undoubtedly belong to our field, especially that a group is linear of degree n over a field if its finitely generated subgroups are linear of degree n over a field (in other words we have a *local* property); in the same paper he considered simultaneous resolution of infinitely many equations over a field and compared solubility of a finite system of equations over a field of characteristic 0 and over fields of finite characteristic, which anticipates one of the themes of more advanced research. The paper does not contain a single word of logic, probably because in 1940 Maltsev did not completely control the situation and lacked a precise formulation of the method of diagrams; instead he used purely algebraic arguments. In his later model-theoretic papers, including [11], he never referred to [10], with the rather curious result that some of Maltsev's most spectacular achievements in model-theoretic algebra, although found in every book discussing linear groups, are ignored by most model theorists and are omitted in the standard model theoretic bibliographies and compilations [1, 4, 12].

By its own nature, the subject resists systematic treatment. Various books have been devoted to sections of it (cf. [3, 15]). The present book does not try to focus upon the most significant issues nor to give a historical survey. The following extract of the preface gives an adequate idea of its spirit:

"Our notes should by no means be regarded as a textbook, either in model theory or in algebra; in particular, we do not pretend to any kind of completeness. The topics we have treated are selected according to personal taste. A guiding principle has been to omit subjects that have already been treated in textbooks or (well-known) lecture notes."

Of course the above "guiding principle" cannot be a very consistent one: by the time the book had appeared modules had been the subject of an extensive monograph [14] (and Galois groups had been covered in [5]). However, as far as modules are concerned, the present book is more oriented towards the model-theoretic treatment of various homological dimensions. On the whole it is fair to say that a large part of the book has not appeared earlier in book form and that a sizable part is apparently published for the first time. Here is a brief list of the contents:

Chapter 1 is an introduction devoted to ultraproducts, the compactness theorem, Hilbert's Nullstellensatz and Hilbert's seventeenth problem. The logical preliminaries are very modest and are dealt with in the first fifteen pages. Stability theory (in the sense of model theory; stable range conditions appear on p. 271) is absent from the book. Chapter 2 to 5 discuss fields, elementary equivalence of all kinds of rings and fields, polynomial rings, function fields, power series fields, with attention paid to the (Tsen-Lang)

C_i conditions, Hilbertian fields and Galois groups. Rings elementarily equivalent to the ring of integers are called Peano rings and are investigated in Chapter 4 (in most of the results of this chapter it would have been enough to consider rings satisfying the first order Peano axioms, as it is clear from [19]). Modules in various forms dominate the scene from Chapter 6: we have the first order theory of modules over a fixed ring, the two-sorted language of modules over unspecified rings, large doses of homological algebra, model-theoretic study of classes of rings which enter naturally into the picture, representation theory of finite dimensional algebras. The book ends with a list of open problems.

From whom is the book meant? As several chapters start from scratch, it is certainly not unsuitable for some beginners with a reasonable familiarity with algebra. It will certainly be very useful to the experts or to those who wish to investigate from a model-theoretic point of view homological algebra. While the required preliminaries in model theory are minimal, the reader should know a substantial amount of algebra in order to read the last chapters, but no specific knowledge is required to appreciate the scores of nice mysterious theorems found at the beginning of the book of which here is a sample: Theorem 2.38. *An algebraic number field of finite degree over* $\mathbf{Q}$ *is not elementarily equivalent to a pure transcendental extension* $F(T)$ *of any field* F; Theorem 4.23. *Let* K *be a field admitting a unique ordering* (*whether this assumption is needed is open*), *then if* $K(X)$ *and* $\mathbf{Q}(X)$ *are elementarily equivalent the fields* K *and* $\mathbf{Q}$ *are isomorphic.*

While the book will provide inspiration to teachers and students as it goes often into unchartered territory, it is difficult to be more precise about its appeal as it clearly suffers and benefits from an excess of cleverness and a lack of organization. A few examples will suffice:

Two different proofs are provided for the Nullstellensatz for an algebraically closed field by separating the countable and uncountable case, while each case is a consequence of the other one. Theorem 2.3 as stated implies only that there exists an ultrafilter on $\mathbf{N}$ yielding $\aleph_1$-saturated ultraproducts while the result is used for every non trivial ultrafilter on $\mathbf{N}$ (cf. p. 24). On p. 49 a subtle proof, based on Roth's theorem, is given of the following result: there exists a real-closed subfield S of the field of real numbers such that the fields $A(X)$ and $S(X)$ are not elementarily equivalent, where A denotes the field of real algebraic numbers. The expert will admire the proof, presumably published for the first time, but the beginner should have been referred to [17], where the same theorem and related questions are treated in a different way and which is omitted in a bibliography of 213 titles, many of which are less relevant than [17]. Flat modules appear on p. 96 and are precisely defined on p. 229. On p. 192 the fact that a commutative Noetherian ring is algebraically compact if and only if it is a direct product of finitely many complete local rings is said to be "well known" but gets a one page proof as Theorem 11.3 on p. 283. The authors investigate as previously said the elementary equivalence of polynomial rings over fields, a subject where surprising results were obtained by one of the authors, but refer only in passing to the paper of Bauval [2] which completely settles the question (p. 268) and yields results more fundamental than Theorem 10.39. To avoid any ambiguity it should be said that the authors are more than generous in giving credit and that they are silent on the novelty of their own methods. I suspect that they did not state Bauval's result because it requires more logic than they provided to the reader, which may pinpoint a fundamental weakness of the book. There may be too much algebra and too little logic in it and one wonders: is this proportion dictated by the choice of topics or the other way around? On the other hand, the reader can easily browse through the book, is not under the obligation to read it in chronological order and is constantly stimulated by interesting proofs and exercises.

There are few misprints and slips, none of which is critical. As clear from the beginning of this review and contrary to what is asserted on p. 158, THE proof of the classical result of Szmielew, stating that the theory of abelian groups is decidable, DOES NOT DEPEND on the classification of algebraically compact groups (but a later proof uses it). I was a bit puzzled by Proposition 6.30 of which here is a literal transcript: "*for finitely presented modules we have: let R be a commutative noetherian ring and S a multiplicatively closed subset of R. If M and N are elementarily equivalent finitely generated R-modules, then the modules of fractions $S^{-1}M$ and $S^{-1}N$ are elementarily equivalent.*" The simplest way to solve the mystery is to replace "finitely presented" with "finitely generated"; another way is to realize that the proposition is true if one takes an arbitrary commutative ring R and finitely presented (or more generally pure-projective) elementarily equivalent

R-modules M and N. A book whose slips are creative is certainly a good book.

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Variational principles for nonpotential operators, by V. M. Filippov. Transl. Math. Monographs, vol. 77, Amer. Math. Soc. Providence, RI, 1989, 239 pp., $99.00. ISBN 0-8218-4529-2

Variational methods reduce the problem of solving linear or nonlinear operator equations to an equivalent problem of determining the stationary values of a suitable functional. If a functional can be found such that its derivative, in some sense, is the operator whose equation is to be solved, then the points at which the functional achieves its extreme values coincide with the solutions of the given operator equation. The usefulness of variational methods is based on the fact that in general it is easier to obtain existence and approximation results for the stationary values of a functional than for the solution of operator equations.

Following the publication of Hilbert's fundamental paper *Über das Dirichletsche Prinzip* [5] and the work of Friedrichs [3], Levi [6], Zaremba [12], and others, the development of variational principles and methods for partial differential and abstract equations became widespread and received theoretical justification in mathematical literature. Today there are a number of excellent books and monographs devoted to variational methods for solving abstract and differential equations (i) $A(x) = f$ in suitable Hilbert spaces when A is a densely defined symmetric operator if (i) is a linear equation, or A is a **potential operator** (i.e. its Gateaux derivative is symmetric) if (i) is a nonlinear equation. However, there does not appear to be a single book devoted exclusively to the variational principle concerning the solvability of equations (i) when A is nonsymmetric (if linear) or nonpotential (if nonlinear), although there is a large number of research papers dealing with variational approach to specific equations, scattered over various mathematics and physics journals and university reports, some of