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The subgroup structure of the finite classical groups, by Peter Kleidman and Martin Liebeck. London Mathematical Society Lecture Note Series, vol. 129, Cambridge University Press, Cambridge, 1990, vii + 303 pp., $29.95. ISBN 0-521-35949-X

There are many questions to ask and answer about subgroups of the classical groups. The questions considered in *The subgroup structure of the finite classical groups* have their roots in the theory of permutation groups. The focus of the book is the central problem of modern permutation group theory: Describe the maximal subgroups of the finite simple groups.

To get some feeling for the questions the authors consider, let us imagine that we are permutation group theorists faced with a problem involving permutation groups. Recall first that each transitive permutation representation of a group G on a set X is equivalent to a representation of G on the cosets of some subgroup H by right multiplication. Indeed H is the stabilizer in G of a point of X. Further as permutation group theorists, we are used to reducing our questions to a problem about *primitive groups*: groups G preserving no nontrivial partition on X. Finally we know that G is primitive if and only if H is maximal in G.

Before 1980 these reductions probably would not have made a big dent in our problem. But about 1980 two related events occurred. First, Mike O'Nan and Len Scott independently made an important observation that is sufficiently elementary to have been made in the late 19th Century: the structure of a finite primitive permutation group is highly restricted. (cf. [Sc] and [ASc]). As a matter of fact most such groups are so restricted that in many problems the only primitive groups which cannot be easily analyzed are the *almost simple groups* G. That is G has a unique minimal normal subgroup L and L is a nonabelian simple group. Better, if we embed G in the group $\mathrm{Aut}(L)$ of automorphisms of L via conjugation and identify L with its group of inner automorphisms, then we have $L \trianglelefteq G \leq \mathrm{Aut}(L)$.

Thus in many circumstances we are reduced to the study of the maximal subgroups of the almost simple groups. Until 1980 even this would not buy us much, which is probably why Burnside did not prove the O'Nan-Scott theorem. However in 1981 the finite simple groups were classified, and the Classification buys us a lot. That is it buys us a lot *if* we know enough about the maximal subgroups of the simple groups on the list supplied by the Classification.

At this point we need to recall that the nonabelian finite simple groups consist of

the alternating groups of degree n;
the finite simple groups of Lie type;
26 sporadic groups.

Each finite group $X(F)$ of Lie type is a finite dimensional linear group over some finite field F. In most cases $X(F)$ has an analogue $X(\mathbf{C})$ over the complex numbers which is a simple Lie group. Thus, as in the case of Lie groups, the finite groups of Lie type come in two flavors: the *classical groups* and a finite number of families of *exceptional groups.*

It should be possible to completely enumerate the maximal subgroups of the sporadic groups and the exceptional groups. This has not yet been accomplished but it would be surprising if this enumeration were not complete within ten years. On the other hand an enumeration of the maximal subgroups of the classical groups or the alternating groups seems in the former case to be equivalent to an enumeration of the modular irreducibles for the quasisimple groups, and in the latter to an enumeration of the primitive permutation representations for all almost simple groups. In short it is probably not possible to enumerate such subgroups in a precise way. Nevertheless it is a curious fact that for the moment the available description of the subgroup structure of the alternating and classical groups is in many ways better than that for the sporadic and exceptional groups. We will see why this is so when we learn more about the Kleidman-Liebeck book.

The subgroup structure of the symmetric group G of degree n is best exhibited via its action on the set X of order n. In particular it is a consequence of the O'Nan-Scott theorem that if $H \leq G$ then either H stabilizes one of several well known "natural structures" on X or H is almost simple and primitive on X (cf. [A2]). The "natural structures" are substructures, coproduct structures, product structures, affine structures, and diagonal structures. Thus each maximal subgroup of G is either the stabilizer of one of our structures or a primitive almost simple group.

To determine all maximals it remains to find the inclusions among such subgroups and enumerate all primitive permutation representations of the almost simple groups. As I have suggested, the latter task is probably impossible. Liebeck, Prager, and Saxl accomplished the former in [LPS]; they show that, modulo an explicit list of exceptions, all of our candidates are maximal. In particular almost always if $\pi: L \to G$ is a primitive permutation representation of a finite simple group L on X then the stabilizer M of the equivalence class of π is a maximal subgroup of G. So in a weak sense we have determined the maximal subgroups of the symmetric and alternating groups.

The book under review does something similar for the classical groups. Recall that the classical groups are the finite dimensional general linear groups, orthogonal groups, symplectic groups, and unitary groups. That is a classical group G over a field F is the isometry group of a trivial or nondegenerate symmetric, alternating, or unitary form f on some finite dimensional vector space V over F. The object $X = (V, f)$ exhibits the subgroup structure of its automorphism group G and plays the role that the set of order n played in our analysis of the symmetric group.

While the problem of enumerating the subgroups of the exceptional groups and sporadic groups is a finite problem in that these groups are of bounded dimension, the objects these groups act on are not as nice as the n-set or the space (V, f). So for the moment our knowledge of the subgroup structure of these groups is sometimes less detailed than our knowledge of the subgroups of the symmetric groups and classical groups.

The analogue of the O'Nan-Scott theorem for the finite classical groups appears in [A1]; it says that if H is a subgroup of G then either H preserves some "natural structure" on X or the image of H in the projective group PG is almost simple and irreducible on V. Thus H is the normalizer of a *quasisimple subgroup* L of G (i.e. L is a perfect central extension of a nonabelian simple group) and L acts absolutely irreducibly on V with the representation primitive, writable over no proper subfield of F, and tensor indecomposable. Let us term such a representation *super irreducible.* The "natural structures" are substructures, coproduct structures, tensor product structures, and vector space structures over subfields and extension fields of F.

So to determine the maximal subgroups of the classical groups we must first determine the inclusions among the stabilizers of our structures and the normalizers of quasisimple super irreducible subgroups L of G. Actually we want more; namely PG is almost simple with simple normal subgroup L and we need to answer the same question for each group $\overline{G}$ between L and PG.

The Main Theorem of the Kleidman-Liebeck book proves that if $\dim(V) \geq 13$ then the stabilizer in $\overline{G}$ of each of our structures is maximal modulo an explicit list of exceptions. Further it describes the action of $\overline{G}$ on these structures and the stabilizer in $\overline{G}$ of each structure.

Thus it remains to study inclusions involving a super irreducible quasimple subgroup of G. One can conjecture that the situation here is the same as for the symmetric group; that is modulo a short explicit list of exceptions, the normalizer in G of each super irreducible quasisimple subgroup L of G is maximal. While we seem to be far from a proof of this assertion, it has been shown to be true by Seitz [Se1, Se2] in the most interesting situation: when L is of Lie type in the same characteristic as F.

Now to the book itself. After a brief introduction, Kleidman and Liebeck begin with a chapter on the basic properties of the finite classical groups. To get the most out of this chapter the reader should probably be familiar with introductory material on the general linear group, its projective geometry, bilinear forms over finite fields, and their geometry, but this background aside the chapter is fairly self contained and supplies a wealth of detailed information about the finite classical groups and their geometries.

Chapter 3 contains the statement of the Main Theorem and includes tables describing in general terms the stabilizers of our structures, the action of G on these structures, and indicating when the stabilizers are not maximal. Chapter 4 contains more detailed information about the structures and their stabilizers.

Chapter 5 contains information about each of the finite simple groups such as the group order, its Schur multiplier, outer automorphism group, etc. There is an extended discussion of the minimal degree of a faithful representation for each group as a permutation group, linear group, or projective linear group in various characteristics. There is a brief discussion of the representation theory of groups of Lie type and characteristic p over fields of characteristic p.

Finally the last few chapters contain the details of the proof of the Main Theorem. These chapters are the most technical and probably of interest only to specialists.

There is a growing group of mathematicians who apply the Classification via appeals to the subgroup structure of the simple groups. In addition to proving a core result facilitating such appeals, "Subgroup Structure" collects in one place many of the tools necessary for applications; e.g. lists of minimal degrees of faithful representations. More important, it contains an extended discussion of the large subgroups of the classical groups (the stabilizers of structures) which recur in applications.

Mathematicians appealing to the Classification can profitably refer to the Kleidman Liebeck book. Those who make such appeals regularly will want to own the book.

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Michael Aschbacher
California Institute of Technology