BOOK REVIEWS

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Asymptotic approximations of integrals, by Roderick Wong. Computer Science and Scientific Computing, Academic Press, London, 1989, viii+543 pp., $69.95. ISBN 0-12-762535-6

The importance of asymptotic approximation in mathematics and in applied sciences, both from the practical and from the theoretical standpoint, is unquestionable. Thus the appearance of a modern treatment discussing the manipulative techniques of this subject and its theoretical basis must be welcomed by all, particularly by those interested in the many developments in applied mathematics, physics, and engineering.

The history of asymptotics, and in particular of the asymptotic expansions of integrals, is as old as mathematical analysis itself, one of the first examples being discussed by Euler (1754). Successive important contributions were made by Laplace (1820) and Stokes (1850). However, the use of asymptotics became systematic only after Poincaré's celebrated definition of asymptotic series (1886). This powerful mathematical tool gives an analytic meaning to the manipulation of a wide class of formal series and constitutes a rigorous basis for methods of asymptotic analysis. Among several other applications, this important definition provided impetus for the development of asymptotic techniques for the evaluation of special integrals which depend on small or large characteristic parameters. Since Poincaré's time these techniques have penetrated several branches of applied mathematics and have, in turn, been affected by developments of this field, acquiring more and more global features. Areas of application include: optic diffraction, crystallography, acoustical scattering, wave propagation in dispersive media, electrical circuit theory, magnetoconvection in electrical conductors, nuclear scattering, theory of probability, statistics, and fluid mechanics. A sample of recent applications can be found in [5].

All integrals treated in this book can be written in the general form

$$I(\lambda) = \int_C F(z;\lambda)\,dz,$$

where C may be taken in context to be either a path in the complex z-plane (Chapters 1 through 7) or a multidimensional real domain (Chapters 8 and 9).

In several practical circumstances it happens that the most significant contribution of the integral depends on the behavior of $F(z;\lambda)$ in a restricted region containing certain special points of C called "critical points." Indeed, an asymptotic approximation of $I(\lambda)$ is generally understood to be an expansion of $I(\lambda)$ valid only in the neighborhood of these critical points. The standard problem is then one of determining such an estimate of $I(\lambda)$. An annotated outline of the work follows.

The first two chapters are a survey of the most commonly used methods for the asymptotic expansion of integrals and contain references for much of this standard background material. The first chapter describes fundamentals on asymptotics including integration by parts, Watson's lemma and the Euler-Maclaurin formula. The next chapter illustrates the well-known Laplace's method for estimating exponentially decaying integrals, a less-known generalization involving expansions with logarithmic scales, the steepest-descent and the saddle-point methods with some of their many variants. The chapter concludes with Darboux's method for obtaining the asymptotic expansion for the coefficients of Maclaurin series. The various methods are explained in detail and illustrated with a selection of examples involving special functions.

Chapter 3 deals with a modern application of the Mellin transform for deriving asymptotic expansions of integrals. This involves re-expressing the integral as the inverse of the product of two Mellin transforms which can be analytically continued into the right half-plane as a meromorphic holomorphic function and then applying the Cauchy residue theorem. This flexible asymptotic technique, which during the last two decades has been systematically developed by Handelsman and Lew, requires some little-known properties of the Mellin transform and proves to be well suited for generating high-order terms in many different situations. Its inclusion in the book is well timed and appropriate.

Chapter 4 is devoted to one of the few known methods for deriving error bound estimates of asymptotic expansions. This makes use of Abel's classical summation formula. Its applications seem, unfortunately, to be limited to oscillating integrals of the Fourier and Hankel type.

The subject treated in the next two chapters is probably the most notable contribution of this book. The appearance of a divergent integral is one of the most serious difficulties occurring in many derivations of asymptotic expansions. In certain cases this obstacle may be circumvented by an attentive use of distribution theory, which may also be used to provide a rigorous estimate of the remainder. This theory is derived in Chapter 5 and applied in Chapter 6. Among other things Chapter 6 includes asymptotic expansions of Stieltjes and Hilbert transforms at infinity and Fourier and Laplace transforms near the origin. The notion of distributional approach is of course not new, but is described piecemeal in the literature; the systematic exposition of this theory in this book is welcome.

Most interesting asymptotic expansions are not uniform. However uniform asymptotic expansions are the subject of Chapter 7, where cases in which singularities (poles or branch points) may coalesce with one or more saddle points are considered. This chapter seems to be devoted almost exclusively to examples, and mainly to one involving asymptotic expansions of Laguerre polynomials.

Finally, Chapters 8 and 9 deal with the derivation of nonuniform asymptotic expansions of several types of multidimensional integrals. As in the one-dimensional case, the principal contribution to the integral is determined from the position of certain critical points of the integrand function. Under reasonable general assumptions, the author examines various situations which may occur in practical problems and derives the corresponding expansions. This part is undoubtedly very difficult for the beginning reader. Difficulties arise from the fact that the techniques developed for one-dimensional integrals cannot be directly extended to the multidimensional case; unfortunately in contrast to the rest of the book, the reader finds only one, somewhat modest, example in two chapters. The part of asymptotics treated in these two chapters is a topical subject: basics and methods of sufficiently general relevance to allow for future applications are developed.

On the whole this book is carefully written and the author's intent is clearly to communicate all aspects of the subject. The standard mode of presentation is followed: after having obtained a method (generally well motivated by example), the author states the relevant theorem which assures that the expansion is valid for a reasonable class of integrals. When possible, realistic estimates for the remainder of the truncated expansions are discussed. In these cases the treatment follows the line of Olver's book [3]: it is concerned with both pure asymptotic and error analysis and it emphasizes all aspects of the development and use of asymptotic approximations and expansions.

The treatment is modern in the sense that there is no hesitation in using recent results of analysis when they contribute to the enlargement of the domain of application of the asymptotic method. This is especially true for Chapters 5, 6, 8, and 9. The necessary background is included for the benefit of uninitiated readers.

This book may be intended as a guide for those already familiar with the subject or as a text for the uninitiated. For the first four chapters (about half the book) only basic mathematical prerequisites are assumed: undergraduate level courses in advanced calculus and a course on complex variable theory should be sufficient. The numerous worked examples allow the reader to follow the procedure step-by-step and so master the methods. Things are not so simple in the remaining chapters. Here difficulties arise from the contribution of various results of mathematical analysis and of theory of special functions. However, the diligent reader should be able to fill the gaps by himself.

This book has a suggestive collection of elaborated examples and a stimulating list of exercise (with a few hints) which complete and extend the scope of the treatment of the subjects. Each chapter ends with useful notes regarding the referenced papers.

This book is not lacking in defects. In the choice of subjects the influence of the author's personal preference and experience, which is natural in almost all books, in here is sometimes exaggerated. The author's facility with subjects to which he has significantly contributed may have induced him to pass over some details and to overemphasize other results. Chapters 3 and 6 deal with two different methods of approximating integrals. In many cases the two methods have analogous performances: the features seem to be superimposed and the courses interpenetrated. For instance, the expansion of the Laplace transform near the origin, which is derived in Chapter 6 using distribution theory, can be also obtained by means of the Mellin transform [2]. A brief discussion emphasizing unlikeness and interconnections between the two methods would have been of help to the casual peruser (note that [2] in the book is not mentioned in this context). As in most treatises in asymptotics, this book uses special functions as a collection of aesthetically elegant illustrative examples. However, this topic, which has generally little academic status in the mathematical curriculum, plays an important role in uniform asymptotic expansions and thus one may not leave their specific properties out of discussion. The asymptotic expansions of Laguerre polynomials, which in the book are derived from integrals, were originally obtained in a much simpler way by Tricomi [4] and by Erdélyi [1] using differential equations.

There are also some slips of the pen: the Riemann-Lebesgue lemma, which applies to absolutely convergent integrals is used twice without proof and references (page 16, page 119), and is proved only on page 200; the lemma is proved for integrals of continuous functions and is presented as "a generalization" of one given in [3, page 76], but the latter applies to integrals of sectionalby continuous functions. Some references are superabundant (see, for instance, the formula for reversion of series on pages 50, 82, and 90); other references are omitted in the list (Wong, 1982, on page 240), and further references appear in the wrong context (Gabutti, 1982). Some misprints and typographical errors are of a trivial nature (like an omitted index).

All these defects are minor. This book is a good one and is a welcome addition to the literature. It presents many of the most important methods and provides an intelligent and useful discussion of them for a wide mathematical audience, without compromising mathematical rigor. In fact, even if this book is written in the spirit of a mathematician, it is also, perhaps mainly, devoted to users of mathematics in physics and engineering, offering a source of newly acquired information which they might need in the asymptotic approximation of integrals. Since this is also the author's understanding stated in the preface, this reviewer feels that the book has been successful in its aim.

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Generatingfunctionology, by Herbert S. Wilf. Academic Press, Harcourt Brace Jovanovich, 182 pp., $29.95. ISBN 0-12-751955-6

I suppose that all of us had at some stage to struggle with poker hands. What is the chance of a hand of five cards containing a run of four red cards? Having gotten the answer, one was never quite sure that it was right and could easily be persuaded by a more expert friend that it should be something quite different.

If a topologist is a man who knows the difference between an orange and a bicycle tire, then a combinatoric is a man who can find the probability of a poker hand in 1-minute flat. The problems dealt with in this book are at first easy but later become quite a bit harder than the above. Consider the following from p. 65:

Definition. A card $C(S, p)$ is a pair consisting of a finite set S (the label set) of positive integers and a picture $p \in P$. The weight of C is $n = |S|$. A card of weight n is called standard if its label set is $[n] = \{1, 2, \dots, n\}$. A hand is a set of cards whose label sets form a partition of $[n]$ for some n. The weight of a hand is the sum of the weights of the cards in the hand. A deck $\mathscr{D}$ is a finite set of standard cards, whose weights are all the same and whose pictures are all different. An exponential family $\mathscr{F}$ is