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A CLASSIFICATION OF COHERENT STATE REPRESENTATIONS OF UNIMODULAR LIE GROUPS

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1. Introduction

Let G be a connected Lie group and $(\pi, \mathscr{H})$ a unitary representation of G on a complex Hilbert space $\mathscr{H}$. Throughout we shall assume that $(\pi, \mathscr{H})$ is *nontrivial* in the sense that $\dim \mathscr{H} > 1$. By a *coherent state orbit* (CS orbit for short) for $(\pi, \mathscr{H})$ we mean a complex orbit of G on the projective space $\mathbf{P}(\mathscr{H})$ (which is equipped with a natural structure of an (infinite-dimensional in general) Kaehler manifold (cf. [L])). We call $(\pi, \mathscr{H})$ a *coherent state representation* (CS representation for short) if (1) it admits a CS orbit, (2) is irreducible and (3) has (at most) discrete kernel, and we call G a CS group if it possesses CS representations. The purpose of this note is to announce a complete classification of connected unimodular CS groups and their CS representations (Theorems 1 and 2 below). This generalizes the results of Enright-Howe-Wallach [EHW] and Jakobsen [J] on the classification of unitary highest weight (or holomorphic) representations of reductive groups (which coincide with the CS representations as we have shown in [L]). The proofs are "geometric," the main tool being the recent structure theory of homogeneous Kaehler manifolds due to Dorfmeister and Nakajima [DN].

In physics, any orbit on $\mathbf{P}(\mathscr{H})$ is called a system of coherent states in the sense of Perelomov (see [P] and the references therein).

Of particular importance are symplectic coherent state orbits; in many cases such an orbit may be interpreted as the classical phase space of the system whose quantum phase space is $\mathbf{P}(\mathscr{H})$. Such an embedding of the classical phase space into the quantum one is the starting point of Berezin's quantization (see [B1] and [B2];

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see also [T] for a comparison of Berezin's quantization with the Kostant-Souriau geometric quantization) and the "quantization of states" proposed recently by Odzijewicz (see [O1] and [O2]). In both theories, the case of complex orbits plays a distinguished role. On one hand, "complex" coherent states are in a sense closest to the classical states [P] and on the other, we may apply in this case powerful techniques of complex analysis (with Bergman type reproducing kernels playing an essential role).

Thus there is a strong physical motivation for studying CS representations.

2. Basic properties of CS representations

Here the term CS representation refers to a $(\pi, \mathscr{H})$ which has property (1) but not necessarily (2) and (3).

Proposition 1 [L]. *Any* CS *orbit has a natural structure of a Hamiltonian G-space and the corresponding moment mapping takes it diffeomorphically onto an integral coadjoint orbit with Kaehler* (*i.e. positive totally complex*) *polarization.*

There is a natural holomorphic line bundle $\mathbf{E}$ over $\mathbf{P}(\mathscr{H})$ whose fiber at $[v] = \mathbf{C}v$ is the dual $[v]^*$. The linear dual $\mathscr{H}^*$ of $\mathscr{H}$ is naturally isomorphic to the space of holomorphic sections of $\mathbf{E}$. Given a CS orbit $G \cdot [v]$ corresponding to a CS representation $(\pi, \mathscr{H})$, we get a natural map from $\mathscr{H}^*$ to the space $\Gamma(G \cdot [v], \mathbf{L})$ of holomorphic sections of $\mathbf{L}$, the restriction of $\mathbf{E}$ to $G \cdot [v]$.

Proposition 2. *The following are equivalent.*

(i) v *is a cyclic vector for* $(\pi, \mathscr{H})$.

(ii) *The map* $\mathscr{H}^* \to \Gamma(G \cdot [v], \mathbf{L})$ *is injective.*

(iii) $(\pi, \mathscr{H})$ *is irreducible.*

The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are clear, and (ii) $\Rightarrow$ (iii) can be deduced from a well-known theorem of Kobayashi [K].

3. Three special cases

It turns out that the case of a general unimodular group can be reduced to three special cases, which we shall now briefly discuss.

3.1. **Heisenberg groups.** Let H_n be a $(2n + 1)$-dimensional Heisenberg group (not necessarily simply connected). Identify the (multiplicative) group $\mathrm{X}(C)$ of unitary characters of the center C of H_n with an (additive) subgroup of the dual c^* of the Lie algebra of C. The infinite-dimensional irreducible unitary representations of H_n are in 1-1 correspondence with the nonzero elements λ of $X(C)$, $(\beta_\lambda, \mathscr{F}_\lambda)$ being the unique (up to equivalence) representation with λ as central character (or, in other terms, the unique representation corresponding, via Kirillov's bijection, to the integral coadjoint orbit $\mathscr{O}_\lambda$ determined by λ). It is well known that any $(\beta_\lambda, \mathscr{F}_\lambda)$ is a CS representation. Any of the CS orbits on $\mathbf{P}(\mathscr{F}_\lambda)$ is mapped by its moment onto $\mathscr{O}_{-\lambda}$. This establishes a 1-1 correspondence between these orbits and Kaehler polarizations of $\mathscr{O}_{-\lambda}$ which, in turn, are in 1-1 correspondence with points of the Siegel space $\mathfrak{S}_n$ (i.e. the Hermitian symmetric space $\mathrm{Sp}(2n, \mathbf{R})/\mathrm{U}(n)$).

Next we consider reductive groups. We shall say that a reductive group is of *compact* (resp. *noncompact*) *type* if its Lie algebra is so.

3.2. **Groups of compact type** [KS]. Any such group is a CS group and any of its nontrivial representations is a CS representation. For any CS representation, there is exactly one CS orbit, namely the orbit through a highest weight line. Geometrically, these orbits are compact simply connected homogeneous Kaehler manifolds (i.e. flag manifolds).

3.3. **Groups of noncompact type** [L]. Such a group is a CS group if and only if it is of *Hermitian type* (i.e. the symmetric space $\mathscr{D}$ associated with it is of Hermitian type). CS representations are the highest weight representations. Again the orbit through a highest weight line is the unique CS orbit for a given CS representation. Geometrically, it is a holomorphic fiber bundle over $\mathscr{D}$ (equipped with one of its invariant complex structures) with flag manifolds as fibers.

4. Homogeneous Kaehler manifolds

Our approach to the problem of classifying CS groups is based on Dorfmeister-Nakajima theorem [DN] (which gives an affirmative answer to a long standing conjecture of Vinberg and Gindikin). For our purposes, it is convenient to state it as follows. *Every homogeneous Kaehler manifold X has a holomorphic double fibration*

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow & \\ M & \rightarrow & \mathscr{D}, \end{array}$$

where M is a homogeneous Kaehler manifold without flat homogeneous Kaehler submanifolds and the fibers of $X \rightarrow M$ are flat *homogeneous Kaehler manifolds (i.e. they are of the form* $\mathbf{C}^n/\Gamma$, *where* Γ *is a discrete subgroup of* $\mathbf{C}^n$ *and the Kaehler metric is induced by the standard Kaehler metric on* $\mathbf{C}^n$), $\mathscr{D}$ *is a homogeneous bounded domain and the fibers of* $M \to \mathscr{D}$ *are flag manifolds.* Such a double fibration is unique and is preserved by all automorphisms of X.

5. Structure of a CS orbit

Now suppose $(\pi, \mathscr{H})$ is a CS representation of G and $X = G \cdot [v] \subset \mathrm{P}(\mathscr{H})$ is a CS orbit such that neither its flat fibers nor $\mathscr{D}$ reduce to points. The fact that X is a Hamiltonian G-space implies that these flat fibers are isomorphic to some $\mathbf{C}^n$ and coincide with the orbits of a Heisenberg group N (of dimension $2n+1$) which is contained in G as a normal subgroup. Let J_N denote the moment mapping of the corresponding Hamiltonian action of N on X. Since the orbits of this action are symplectic, the symplectic reduction theorem (see [AM]) implies that $J_N(X)$ is a single coadjoint orbit $\mathscr{O}_\lambda$.

N being a normal subgroup of G, there is a homomorphism

$$\tilde{\rho}: G \to \mathrm{Aut}(N), \; g \mapsto \mathrm{Int}\, g|_N$$

(where $\mathrm{Int}\, g$ denotes the inner automorphism of G corresponding to g), which factors through N to give a homomorphism

$$\rho: S = G/N \to \mathrm{Out}\, N = \mathrm{Aut}\, N/\mathrm{Int}\, N.$$

It is clear that $\tilde{\rho}(S) \subset (\mathrm{Aut}\, N)_\lambda$, the stabilizer of $\mathscr{O}_\lambda$ (or λ) in $\mathrm{Aut}\, N$ (which is the same for all $\lambda \neq 0$), and, consequently, $\rho(S) \subset (\mathrm{Aut}\, N)_\lambda/\mathrm{Int}\, N \cong \mathrm{Sp}(2n, \mathbf{R})$.

Being a complex submanifold of X, each N-orbit carries a Kaehler polarization which is mapped by J_N into a Kaehler polarization of $\mathscr{O}_\lambda$. We thus get a ρ-equivariant map from the orbit space $M = X/N$ to the space of all Kaehler polarizations of $\mathscr{O}_\lambda$, i.e. the Siegel space $\mathfrak{S}_n$. It can be shown that this map is holomorphic. Hence it factors through the compact fibers of M to give a ρ-equivariant holomorphic map

$$\rho_{\mathscr{D}}: \mathscr{D} \to \mathfrak{S}_n.$$

6. Classification of unimodular CS groups

From now on we assume that G is unimodular (and nonreductive). Using the results of the preceding section it is not hard to show that then $S = G/N$ is also unimodular and so is its quotient $S/N_{\mathscr{D}}$ which acts effectively on $\mathscr{D}$. Moreover, $N_{\mathscr{D}}$ is of compact type (here the assumption that π has discrete kernel is essential). Now a theorem of Hano [Ha] asserts that if a unimodular Lie group acts effectively and transitively on a bounded domain, then it is semisimple and the domain is symmetric. Thus $S/N_{\mathscr{D}}$ is semisimple and, consequently, S is reductive and of Hermitian type. It follows that N coincides with the nilradical (the maximal connected nilpotent normal Lie subgroup) of G.

We have sketched the proof of the "only if part" of the following.

Theorem 1. *A connected unimodular (nonreductive) Lie group G is a* CS *group if and only if it satisfies the following conditions.*

(i) *The nilradical N of G is isomorphic to a Heisenberg group H_n.*
(ii) *The reductive group $S = G/N$ is either of compact or of Hermitian type and its image under the natural homomorphism $\rho: S \to \mathrm{Out}(N)$ is contained in* $\mathrm{Sp}(2n, \mathbf{R})$.
(iii) *If S is of Hermitian type, there exists a ρ-equivariant holomorphic map from the Hermitian symmetric space $\mathscr{D}$ associated with S to the Siegel space $\mathfrak{S}_n$.*

That this theorem really classifies unimodular CS groups follows from the results of Satake (see [S2]) who classified ρ-equivariant holomorphic maps $\mathscr{D} \to \mathfrak{S}_n$ (this classification is closely related to the classification of Howe's reductive dual pairs in $\mathrm{Sp}(2n, \mathbf{R})$ (cf. [Ho]).

7. Classification of CS representations

Irreducible unitary representations of the groups which occur in Theorem 1 have been classified by Satake [S1]. Using his results and the results of the preceding sections (with Proposition 2 playing an essential role) we can complete the proof of Theorem 1 and also prove the following.

Theorem 2. *Suppose G has properties* (i)–(iii) *of Theorem* 1. *For any nonzero $\lambda \in \mathrm{X}(C)$, let $(\sigma_\lambda, \mathscr{F}_\lambda)$ be a projective representation of G obtained by composing the (projective) metaplectic representation of* $(\mathrm{Aut}\ N)_\lambda$ *(associated with $(\beta_\lambda, \mathscr{F}_\lambda)$) with $\tilde{\rho}$ and let α be its cocycle (α does not depend on λ and can be considered as a cocycle on $S = G/N$). Let $(\pi_1, \mathscr{E})$ be an irreducible projective* *unitary representation of S with the following properties:*

(i) *its cocycle is* α^{-1};

(ii) *its kernel* $\ker \pi_1$ *is contained in* $N_{\mathscr{D}}$ (*cf.* §6);

(iii) *the corresponding representation of* $S/\ker \pi_1$ *is a* (*projective*) CS *representation.*

Then $(\pi, \mathscr{H})$, *where* $\mathscr{H} = \mathscr{E} \otimes \mathscr{F}_\lambda$ (*Hilbert tensor product*) *and*

$$\pi(g) = \tilde{\pi}_1(g) \otimes \sigma_\lambda(g) \quad \text{for } g \in G,$$

$\tilde{\pi}_1$ *being the composition of* π_1 *and the projection* $G \to S$, *is a* (*linear*) CS *representation of* G *and any* CS *representation of* G *is of this form.*

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