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PROOF OF THE PAYNE-PÓLYA-WEINBERGER CONJECTURE

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In 1955 and 1956 Payne, Pólya, and Weinberger considered the problem of bounding ratios of eigenvalues for homogeneous membranes of arbitrary shape [PPW1, PPW2]. Among other things, they showed that the ratio λ_2/λ_1 of the first two eigenvalues was less than or equal to 3 and went on to conjecture that the optimal upper bound for λ_2/λ_1 was its value for the disk, approximately 2.539. It is this conjecture which we establish below.

Since 1956 various authors have attempted to prove the conjecture of Payne, Pólya, and Weinberger and some have been able to improve upon the constant 3. Specifically, Brands [Br] in 1964 obtained the value 2.686, de Vries [dV] in 1967 obtained 2.658, and Chiti [Ch2] in 1983 obtained 2.586. In addition, Thompson [Th] gave the natural extension of the PPW argument to dimension n, obtaining

$$\lambda_2/\lambda_1 \leq 1 + 4/n \tag{1}$$

as the bound for the analogous problem (eigenvalues of the Dirichlet Laplacian on a bounded domain in $\mathbf{R}^n$) and made the natural

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conjecture that the optimal bound is the value of λ_2/λ_1 for a ball in $\mathbf{R}^n$. This more general conjecture can also be established using the approach we present below. However, to keep the discussion concise we shall restrict it almost entirely to our proof of the original (two-dimensional) PPW conjecture. Only in our concluding remarks do we briefly return to a discussion of the situation for general dimension n.

To be precise, the problem that we consider is the eigenvalue problem for the Dirichlet Laplacian on a bounded domain $\Omega \subset \mathbf{R}^2$:

$$-\Delta u = \lambda u \quad \text{on } \Omega \tag{2}$$

with boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{3}$$

It is well known that the spectrum of this problem is $\{\lambda_i\}_{i=1}^\infty$ where the eigenvalues λ_i are repeated according to their multiplicities (each of which is finite) and

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \lambda_4 \le \cdots. \tag{4}$$

A corresponding sequence of orthonormal eigenfunctions will be denoted $\{u_i\}_{i=1}^\infty$; each u_i is a normalized eigenfunction for the eigenvalue λ_i. Our result may now be stated formally as

Theorem (Payne-Pólya-Weinberger conjecture). *The ratio of the first two eigenvalues of the Dirichlet Laplacian on a domain $\Omega \subset \mathbf{R}^2$, as defined by* (2) *and* (3) *above, satisfies*

$$\frac{\lambda_2}{\lambda_1} \le \left.\frac{\lambda_2}{\lambda_1}\right|_{\Omega=\text{disk}} = \left(\frac{j_{1,1}}{j_{0,1}}\right)^2 \approx 2.539. \tag{5}$$

Moreover, equality obtains if and only if Ω is a disk.

The quantities $j_{0,1}$ and $j_{1,1}$ occurring here denote the first positive zeros of the Bessel functions $J_0(x)$ and $J_1(x)$, respectively. In general $j_{p,k}$ will denote the kth positive zero of $J_p(x)$. Note that no mention need be made of the size of our domains Ω since eigenvalue ratios are unaffected by changes of scale; only shapes matter in these considerations.

We turn now to the proof of our theorem. There are several elements that go into the proof, some of which will be indicated only in broad form here. In particular, we make use of the Rayleigh-Ritz inequality for λ_2, spherical rearrangements, a comparison result of Chiti [Ch1, Ch2], and special properties of certain combinations of Bessel functions. We also require carefully chosen trial functions for u_2 and a topological argument which makes possible our use of the Rayleigh-Ritz inequality.

Starting from the Rayleigh-Ritz inequality for λ_2,

(6)
$$\lambda_2 \leq \frac{-\int_\Omega Pu_1\Delta(Pu_1)\,dA}{\int_\Omega P^2u_1^2\,dA} \quad \text{provided } \int_\Omega Pu_1^2\,dA = 0 \text{ and } P \not\equiv 0,$$

the gap estimate,

(7)
$$\lambda_2 - \lambda_1 \leq \frac{\int_\Omega |\nabla P|^2u_1^2\,dA}{\int_\Omega P^2u_1^2\,dA} \quad \text{provided } \int_\Omega Pu_1^2\,dA = 0 \text{ and } P \not\equiv 0,$$

follows easily by integration by parts. Here dA denotes the standard area element in $\mathbf{R}^2$. We shall actually use two different trial functions P in (7). We take them as

$$P_i = g(r)\frac{x_i}{r}, \qquad i = 1, 2 \tag{8}$$

where g is a nonnegative and nontrivial function of the radial variable r (to be chosen explicitly later) and the x_i are the standard Cartesian coordinates. Obviously, the side condition $P \not\equiv 0$ is satisfied for each of these. To see that we can insure that the other side conditions, $\int_\Omega P_iu_1^2\,dA = 0$ for $i = 1, 2$, can be simultaneously satisfied requires a brief topological argument (from Weinberger [W]). One considers the vector field

$$\vec{v} = \int_\Omega g(r)\frac{\vec{x}}{r}u_1^2\,dA \tag{9}$$

as a function of where we place the origin. Letting D be a ball such that $\Omega \subset D$, it is clear that $\vec{v}$ is defined and continuous on D and since $g(r)u_1^2 \geq 0$ and is supported inside D it follows that on the boundary of D $\vec{v}$ must always point inward. It now follows from the Brouwer fixed point theorem that $\vec{v}$ must vanish somewhere within D. Taking some such point (which is necessarily inside the convex hull of Ω) as our origin guarantees that the side conditions, $\int_\Omega P_iu_1^2\,dA = 0$ for $i = 1, 2$, are both satisfied.

Proceeding with the argument, we now have

$$(\lambda_2 - \lambda_1)\int_\Omega P_i^2u_1^2\,dA \leq \int_\Omega |\nabla P_i|^2u_1^2\,dA \quad \text{for } i = 1, 2 \tag{10}$$

and summing on i yields

$$\lambda_2 - \lambda_1 \le \frac{\int_\Omega [g'(r)^2 + \frac{1}{r^2} g(r)^2] u_1^2 \, dA}{\int_\Omega g(r)^2 u_1^2 \, dA}. \tag{11}$$

This is a key estimate which we call the basic gap inequality. (In his paper [Ch2], Chiti used (11) with the function $g(r) = r$.) A crucial observation is that by summing on i in (10) all angular dependence due to the trial functions has disappeared in (11). This makes possible a sharp use of rearrangement inequalities, which we now discuss.

For an integrable real-valued function f defined on a bounded domain $\Omega \subset \mathbf{R}^n$ the spherical decreasing rearrangement of f (also known as symmetric decreasing rearrangement [BLL, L1, L2, HLP, pp. 276–278] or Schwarz symmetrization [Ba, p. 47, Ka, pp. 15–16]; for more details, see [Ba, HLP, Ka]) is a function, denoted f^* which is defined on the ball $\Omega^* \subset \mathbf{R}^n$ of volume $|\Omega|$ and with center at the origin, which is invariant under rotations and nonincreasing with respect to distance from the origin, and which is equimeasurable with f, i.e. the sets $\{x \in \Omega | f(x) \ge t\}$ and $\{x \in \Omega^* | f^*(x) \ge t\}$ have equal measures for all real t. Since f^* is spherically symmetric we shall abuse notation occasionally and write $f^*(r)$ which is to be viewed as a function of the radial variable $r = |x|$ for r between 0 and the radius of Ω^*. The spherical increasing rearrangement of f, denoted f_*, is defined in the same way but with "nondecreasing" replacing "nonincreasing" in the definition above. Alternatively, one could define f_* via $f_* = -(-f)^*$. The results which we shall need for the arguments to follow are the inequality

$$\int_\Omega fg \, dA \le \int_{\Omega^*} f^* g^* \, dA \tag{12}$$

and the equivalent inequality

$$\int_\Omega fg \, dA \ge \int_{\Omega^*} f_* g^* \, dA. \tag{13}$$

Additionally, it should be noted that the operation of taking the spherical decreasing (resp., spherical increasing) rearrangement is dependent on the domain Ω in the following sense: if $f(r)$ is a nonincreasing (resp., nondecreasing) function of r for all $r \in (0, \max_{x\in\Omega} |x|)$ then

$$f(r)^* \le f(r)$$

(resp., $f(r)_* \geq f(r)$) for all r between 0 and the radius r^* of Ω^*, with strict inequality for $r \in (\min_{x \notin \Omega} |x|, r^*)$ if f is strictly monotone there.

Returning to the main line of the argument, observe that to get the optimal bound (5) we must take $g(r)$ so that (11) becomes an equality if Ω is a ball. Thus we want $g(r)(x_i/r)u_1$ to reduce to an eigenfunction corresponding to λ_2 for $i = 1, 2$ in that case and this motivates the choice

$$g(r) = w(\gamma r) \tag{14}$$

where

$$w(x) \equiv \begin{cases} \dfrac{J_1(\beta x)}{J_0(\alpha x)} & \text{for } 0 \leq x < 1, \\ w(1) \equiv \lim_{x \to 1^-} w(x) & \text{for } x \geq 1, \end{cases} \tag{15}$$

with $\alpha = j_{0,1}$, $\beta = j_{1,1}$, and $\gamma = \sqrt{\lambda_1}/\alpha$. (The function w defined in this way is continuously differentiable on $[0, \infty)$.) Substituting this into the basic gap inequality yields

$$\lambda_2 - \lambda_1 \leq \gamma^2 \frac{\int_\Omega B(\gamma r) u_1^2 \, dA}{\int_\Omega w(\gamma r)^2 u_1^2 \, dA} = \frac{\lambda_1}{\alpha^2} \frac{\int_\Omega B(\gamma r) u_1^2 \, dA}{\int_\Omega w(\gamma r)^2 u_1^2 \, dA} \tag{16}$$

where

$$B(x) \equiv w'(x)^2 + \frac{1}{x^2} w(x)^2. \tag{17}$$

Using properties of Bessel functions one can show

(a) $w(x)$ is increasing on $\mathbf{R}^+$,
(b) $B(x)$ is decreasing on $\mathbf{R}^+$.

Both facts are obvious for $x \geq 1$; the remainder of the arguments will be sketched after we have completed the main argument of the proof.

Using rearrangements one then has

$$\int_\Omega B(\gamma r) u_1^2 \, dA \leq \int_{\Omega^*} B(\gamma r)^* u_1^{*2} \, dA \leq \int_{\Omega^*} B(\gamma r) u_1^{*2} \, dA \tag{18}$$

and

$$\int_\Omega w(\gamma r)^2 u_1^2 \, dA \geq \int_{\Omega^*} w(\gamma r)_*^2 u_1^{*2} \, dA \geq \int_{\Omega^*} w(\gamma r)^2 u_1^{*2} \, dA \tag{19}$$

where $\Omega^* \subset \mathbf{R}^2$ denotes the ball of measure $|\Omega|$ with center at the origin.

To finish proving the PPW conjecture one needs the following comparison result due to Chiti [Ch1]: let S_1 be the ball such that the Dirichlet problem

$$\begin{cases} -\Delta z = \lambda z & \text{in } S_1, \\ z = 0 & \text{on } \partial S_1, \end{cases} \tag{20}$$

has λ_1 as its first eigenvalue and suppose that z is normalized so that

$$\int_\Omega u_1^2 \, dA = \int_{S_1} z^2 \, dA.$$

Then $|S_1| \le |\Omega|$ (with equality if and only if Ω is a ball; this follows immediately from the Faber-Krahn inequality) and there is a point $r_1 \in (0, 1/\gamma)$ such that

$$\begin{cases} u_1^*(r) \le z(r) & \text{for } r \in [0, r_1], \\ u_1^*(r) \ge z(r) & \text{for } r \in [r_1, 1/\gamma]. \end{cases} \tag{21}$$

To be more explicit, one has

$$\int_\Omega u_1^2 \, dA = \int_{\Omega^*} u_1^{*2} \, dA = \int_{S_1} z^2 \, dA,$$

$$z(r) = cJ_0(\sqrt{\lambda_1} r),$$

and

$$S_1 = \{x \in \mathbf{R}^2 \,|\, |x| \le \alpha/\sqrt{\lambda_1} = 1/\gamma\} \subset \Omega^*.$$

From this it follows that if $f(r)$ is an increasing function then, with R denoting the radius of Ω^*,

$$\begin{aligned}
&\int_{S_1} f(r) z^2 \, dA - \int_{\Omega^*} f(r) u_1^{*2} \, dA \\
&= 2\pi \left[\int_0^{r_1} f(r)(z^2 - u_1^{*2}) r \, dr + \int_{r_1}^{1/\gamma} f(r)(z^2 - u_1^{*2}) r \, dr \right. \\
&\qquad \left. - \int_{1/\gamma}^R f(r) u_1^{*2} r \, dr \right] \\
&\le 2\pi \left[f(r_1) \int_0^{r_1} (z^2 - u_1^{*2}) r \, dr + f(r_1) \int_{r_1}^{1/\gamma} (z^2 - u_1^{*2}) r \, dr \right. \\
&\qquad \left. - f(r_1) \int_{1/\gamma}^R u_1^{*2} r \, dr \right] \\
&= f(r_1) \left[\int_{S_1} z^2 \, dA - \int_{\Omega^*} u_1^{*2} \, dA \right] = 0
\end{aligned}$$

so that

$$\int_{\Omega^*} f(r) u_1^{*2}\, dA \geq \int_{S_1} f(r) z^2\, dA \quad \text{if } f \text{ is increasing.} \tag{22}$$

Similarly it follows that the reverse inequality holds if f is decreasing. Using these results and properties (a) and (b), it follows that

$$\int_{\Omega^*} B(\gamma r) u_1^{*2}\, dA \leq \int_{S_1} B(\gamma r) z^2\, dA \tag{23}$$

and

$$\int_{\Omega^*} w(\gamma r)^2 u_1^{*2}\, dA \geq \int_{S_1} w(\gamma r)^2 z^2\, dA \tag{24}$$

and combining these inequalities with (16), (18), and (19) we obtain

$$\begin{aligned} \lambda_2 - \lambda_1 &\leq \frac{\lambda_1}{\alpha^2} \frac{\int_{S_1} B(\gamma r) z^2\, dA}{\int_{S_1} w(\gamma r)^2 z^2\, dA} = \frac{\lambda_1}{\alpha^2} \frac{\int_0^1 B(r) J_0^2(\alpha r) r\, dr}{\int_0^1 w(r)^2 J_0^2(\alpha r) r\, dr} \\ &= \frac{\lambda_1}{\alpha^2} [(\lambda_2 - \lambda_1) \text{ for the ball of radius } 1] \\ &= \frac{\lambda_1}{\alpha^2} (\beta^2 - \alpha^2). \end{aligned} \tag{25}$$

$\lambda_2/\lambda_1 \leq \beta^2/\alpha^2$, which is the inequality (5), is immediate. To see that equality occurs if and only if Ω is a ball now follows easily. The "if" part is obvious and the "only if" part follows from any of the following facts (none of which is difficult): (a) (11) is strict unless Ω is a ball, (b) the second inequalities in (18) and (19) are strict unless Ω is a ball, and (c) the inequalities (23) and (24) are strict unless Ω is a ball. This completes the proof aside from the facts (a) and (b) claimed of $w(x)$ and $B(x)$.

The demonstration of the relevant properties of the Bessel functions occurring here is fairly direct if one works with their product representations. To simplify notation set $\alpha_n = j_{0,n}$ and $\beta_n = j_{1,n}$. Then since $y = \sqrt{x} J_p(x)$ satisfies $y'' + [1 - (p^2 - 1/4)/x^2] y = 0$ it follows by Sturm comparison theory that the sequences of differences of zeros have the following properties:

$$\alpha_1, \alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \ldots \text{is increasing (toward } \pi)$$

and

$$\beta_1, \beta_2 - \beta_1, \beta_3 - \beta_2, \ldots \text{is decreasing (toward } \pi).$$

It follows that

$$\frac{\beta_n}{\alpha_n} = \frac{\sum_{i=1}^{n}(\beta_i - \beta_{i-1})}{\sum_{i=1}^{n}(\alpha_i - \alpha_{i-1})} < \frac{n\beta_1}{n\alpha_1} = \frac{\beta_1}{\alpha_1} \tag{26}$$

(we have taken $\alpha_0 = 0 = \beta_0$ here for notational convenience). Introducing the notation $\tilde{\alpha}_n \equiv \alpha_n/\alpha_1$ and $\tilde{\beta}_n \equiv \beta_n/\beta_1$ for the zeros of the scaled Bessel functions $J_0(\alpha x)$ and $J_1(\beta x)$, respectively, it is clear from (26) that

$$\tilde{\beta}_n < \tilde{\alpha}_n \quad \text{for } n = 2, 3, 4, \ldots \tag{27}$$

(recall that $\alpha = \alpha_1$ and $\beta = \beta_1$). This is the key inequality from which all the necessary inequalities follow.

With

$$A(x) \equiv (\ln w)' = \beta\frac{J_1'(\beta x)}{J_1(\beta x)} - \alpha\frac{J_0'(\alpha x)}{J_0(\alpha x)} \tag{28}$$

one finds, using the product representations for J_0 and J_1,

$$A(x) = \frac{1}{x} - 2x\sum_{n=2}^{\infty}\frac{\tilde{\alpha}_n^2 - \tilde{\beta}_n^2}{(x^2-\tilde{\alpha}_n^2)(x^2-\tilde{\beta}_n^2)} < \frac{1}{x} \quad \text{for } x \in (0, 1] \tag{29}$$

and also

(30)

$$A'(x) = \frac{-1}{x^2} - 2\sum_{n=2}^{\infty}\frac{(\tilde{\alpha}_n^2 - \tilde{\beta}_n^2)[\tilde{\alpha}_n^2\tilde{\beta}_n^2 + (\tilde{\alpha}_n^2 + \tilde{\beta}_n^2)x^2 - 3x^4]}{(x^2-\tilde{\alpha}_n^2)^2(x^2-\tilde{\beta}_n^2)^2} < \frac{-1}{x^2}$$
$$\text{for } x \in (0, 1],$$

the final inequality holding by virtue of the fact that the quadratic

$$f_n(t) \equiv \tilde{\alpha}_n^2\tilde{\beta}_n^2 + (\tilde{\alpha}_n^2 + \tilde{\beta}_n^2)t - 3t^2$$

opens downward and has $f_n(0) = \tilde{\alpha}_n^2\tilde{\beta}_n^2 > 0$ and $f_n(1) = \tilde{\alpha}_n^2\tilde{\beta}_n^2 + \tilde{\alpha}_n^2 + \tilde{\beta}_n^2 - 3 > 0$ for $n \geq 2$ since $f_n(1)$ is clearly increasing in n and $f_1(1) = 0$. From (30) it is clear that A is decreasing on $(0, 1]$ and since $A(1) = 0$ it follows that $A(x) > 0$ for $x \in (0, 1)$ (in fact, from (30) the stronger result $A(x) > 1/x - 1$ on $(0, 1)$ follows). This shows property (a), that w is increasing, since $w' = Aw$ and w is clearly nonnegative. To see property (b), that $B(x) = w'(x)^2 + w(x)^2/x^2$ is decreasing on $[0, 1]$, we argue that both $w'(x)$ and $w(x)/x$ are decreasing (and positive) on $[0, 1]$. For $w(x)/x$ observe that

$$(w/x)' = (xw' - w)/x^2 = (A - 1/x)w/x < 0 \quad \text{on } (0, 1]$$

by virtue of (29). For $w'(x)$ observe that

$$w'' = A'w + Aw' = [A' + A^2]w < 0 \quad \text{on } (0, 1]$$

by virtue of (29) and (30), completing the proof.

We conclude with several additional remarks. First, the argument above extends naturally to the n-dimensional generalization of the PPW conjecture. One finds $\lambda_2/\lambda_1 \le (j_{n/2,1}/j_{n/2-1,1})^2$ in that case with equality if and only if Ω is an n-dimensional ball. In particular, the argument goes through almost unchanged in dimension 3 (where now the sequence corresponding to $\{\alpha_i - \alpha_{i-1}\}_{i=1}^{\infty}$ is constantly π and that corresponding to $\{\beta_i - \beta_{i-1}\}_{i=1}^{\infty}$ decreases toward π); however, in higher dimensions the proof of the analog of $\tilde{\alpha}_m > \tilde{\beta}_m$ for $m \ge 2$ is necessarily more involved. One can also obtain the same bound for the eigenvalues of the Schrödinger operator $H = -\Delta + V(x)$ acting on $L^2(\Omega)$ with Dirichlet boundary conditions if $V \ge 0$ on Ω. These results and further generalizations will be developed and discussed in greater detail in a longer version of this paper which will be published elsewhere [AB2]. We had conjectured the Schrödinger operator result in an earlier paper, after we had established its one-dimensional specialization (see [AB1] and references therein). For further background on the significance and consequences of the PPW conjecture we recommend the paper of Hersch [H] as well as his commentary on several of Pólya's papers: specifically, see the comments on papers 202 and 203 in [HR, pp. 519–522]. As should be apparent to anyone familiar with Chiti's paper [Ch2], our proof owes a lot to his general approach and, most importantly, to his comparison theorem. The work of Weinberger in [W] (also discussed in [Ba]) also has several features in common with ours. Finally, we mention that a second conjecture of Payne, Pólya, and Weinberger, that in two dimensions

$$(31) \qquad (\lambda_2 + \lambda_3)/\lambda_1 \le (\lambda_2 + \lambda_3)/\lambda_1|_{\Omega=\text{ball}} = 2\beta^2/\alpha^2 \approx 5.077,$$

remains open. This would be a stronger result than ours but seems correspondingly more difficult to prove. However, we have been able to use our approach above to establish this and its higher dimensional analogs under certain added symmetry conditions. Specifically, (31) holds in two dimensions if Ω has rotational symmetry of order 4. The details of this will appear in [AB2].

NOTE ADDED IN PROOF

We have recently succeeded in proving that for $\Omega \subset \mathbf{R}^n$ $\lambda_4/\lambda_2 < j_{n/2,1}^2/j_{n/2-1,1}^2$ which establishes the next two cases of the further conjecture of Payne, Pólya, and Weinberger that $\lambda_{m+1}/\lambda_m|_\Omega \le \lambda_2/\lambda_1|_{n\text{-dimensional ball}}$ for $m = 1, 2, 3, \ldots$ This result and related material will appear in our forthcoming paper *Isoperimetric bounds for higher eigenvalue ratios for the n-dimensional fixed membrane problem.*

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