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Lectures on hyponormal operators, by M. Martin and M. Putinar. Birkhauser, Boston, 1989, 305 pp., $80.00. ISBN 3-7643-2329-9

A *hyponormal operator* is a bounded operator T on a Hilbert space $\mathscr{H}$ such that $T^*T \geq TT^*$. This rather innocent definition was introduced by Paul Halmos [11] in 1950 and generalizes the concept of a normal operator (where $T^*T = TT^*$). Why not consider the condition $T^*T \leq TT^*$, you ask? In fact, people do; such operators are called *cohyponormal*. (Needless to say, the two theories are related, though one is not a trivial adaptation of the other since many properties do not travel well when taking adjoints.) The important thing is that there is a prominent example of a hyponormal operator, the unilateral shift. If l^2 is the Hilbert space of square summable sequences and T is defined on l^2 by $T(a_0, a_1, \ldots) = (0, a_0, a_1, \ldots)$, then T is called the *unilateral shift* and is the most basic of hyponormal operators.

Normal operators are completely understood. Indeed, it is possible to define a complete set of unitary invariants for normal operators; equivalently, it is possible to give a model for an arbitrary normal operator. Specifically, in the case that the underlying Hilbert space $\mathscr{H}$ is separable, given any compactly supported regular Borel measure μ on the complex plane and a Borel function m defined on $\mathbf{C}$ with values in $\{\infty, 0, 1, 2, \ldots\}$ such that $m = 0$ off the support of μ, there is a canonically associated normal operator $N_{\mu,m}$ and each normal operator is unitarily equivalent to one of these models. That is, for each normal operator N there is a μ, such a function m, and a unitary operator U with $N = U^*N_{\mu,m}U$. Moreover, two such models $N_{\mu,m}$ and $N_{\nu,n}$, are unitarily equivalent if and only if $[\mu] = [\nu]$ (that is, μ and ν are equivalent measures in the sense that they have the same sets of measure 0) and $m = n$ a.e. $[\mu]$. The details of this can be found in [10], §IX.10. In the case of a nonseparable Hilbert space the theory exists but the measure theory becomes more complicated. This is the subject of the last chapter of [12].

In this way, any question concerning normal operators can be reduced to a question concerning measure theory from which an answer can usually be derived. What about operators that are not normal? The general problem is impossible. In fact, even if the Hilbert space is finite dimensional, the answer is unknown. That is, what are necessary and sufficient conditions for two matrices to be unitarily equivalent?

As it turns out the unilateral shift is a well understood nonnormal operator; it is arguably the best understood nonnormal operator on an infinite dimensional space. For example, in [3] the invariant subspaces of the shift are completely determined. (An *invariant subspace* for an operator T is a closed subspace $\mathscr{M}$ of $\mathscr{H}$ such that $T\mathscr{M} \subseteq \mathscr{M}$.) This paper, a milestone in the development of operator theory, was the first instance where complete information about the invariant subspaces of a nonnormal operator was obtained. In addition, the method of proof involved the use of significant results about analytic functions. The consequences of these two facts are still prevalent in operator theory today.

Success breeds success. Paul Halmos began a strategic attack on operator theory by extracting two properties of the shift in [11]. One was the definition of hyponormal operators and the other the idea of a subnormal operator. A *subnormal operator* is one that has a normal extension; every subnormal operator is hyponormal. Sometimes success also breeds other things and there is a plethora of definitions of classes of operators where a prefix or suffix is attached to the word *normal*. Some of these definitions are useful and important and others lack the necessary stock of examples to guarantee their viability. In this review we will concentrate on hyponormal operators, with only occasional references to the theory of subnormal operators.

Any operator T can be written as $T = N \oplus T_1$, where N is a normal operator and T_1 is an operator with no normal direct summand. Such an operator T_1 is said to be *pure*; T is a pure operator if no such nontrivial direct sum decomposition can be found. Many, but not all, questions concerning hyponormal operators can be reduced to the study of pure hyponormal operators. So throughout this review it will be assumed that T is pure.

Write T in its Cartesian decomposition. $T = X + iY$, where X and Y are self-adjoint operators. One of the first important results in the theory of hyponormal operators, due to C. R. Putnam, is the fact that if T is a pure hyponormal operator, then its real and imaginary parts, X and Y, must be absolutely continuous self-adjoint operators [17]. That is, the spectral measures for X and Y must be absolutely continuous with respect to Lebesgue measure on the real line. Thus the Spectral Theorem for self-adjoint operators can be applied to X, and this operator can be represented as a multiplication operator on $L^2[a, b]$ for some interval in $\mathbf{R}$. The operator Y also has such a representation, but on a different L^2 space. Can Y be represented on the same space $L^2[a, b]$ in a way that is intimately connected with the representation of X?

Indeed, a result of Kato [14], though it is not directly related to hyponormal operators, implies that this can be done. About the same time, many authors began investigating hyponormal operators from this perspective [15, 19, 21, 22].

Note that the self-commutator of T, $[T^*, T] \equiv T^*T - TT^*$, can be expressed in terms of its real and imaginary parts X and Y: $[T^*T] = 2i[X, Y] = 2i(XY-YX)$. If T is a pure hyponormal operator and the self-commutator of T is a trace class operator, then when X is represented as a multiplication operator, the operator Y appears as a singular integral operator in a more tractable form than the general case [15, 19, 21, 22].

For a pair of self-adjoint operators X and Y, say that X and Y *almost commute* if their commutator, $[X, Y] \equiv XY - YX$, is a trace class operator. Helton and Howe [13] began a systematic study of almost commuting pairs of self-adjoint operators by introducing the following tracial bilinear form. Let $\mathscr{P}(\mathbf{R}^2)$ be the collection of all polynomials in two variables with complex coefficients. If $p(x, y)$ and $q(x, y)$ are two such polynomials, define

$$\langle p, q\rangle = \operatorname{trace}(i[p(X, Y), q(X, Y)]).$$

The fact that X and Y almost commute implies that this is well defined. They showed that there is a regular Borel signed measure μ on $\mathbf{R}^2$ having compact support such that

$$\langle p, q\rangle = \int J(p, q)\, d\mu,$$

where

$$J(p, q) = \frac{\partial p}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial q}{\partial x}\frac{\partial p}{\partial y}.$$

Later Joel Pincus [15] showed that this measure μ must be absolutely continuous with respect to planar Lebesgue measure λ, and the Radon–Nikodym derivative, $g = 2\pi\, d\mu/d\lambda$, is precisely the principal function he had studied in conjunction with almost commuting pairs.

This was the starting point for a sequence of papers by Carey and Pincus [6–9]. In particular, in [7] they associate with a hyponormal operator an operator-valued function called the *mosaic* of T. This mosaic turns out to be a complete unitary invariant for pure hyponormal operators with trace class self-commutator. They also showed that for any mosaic the corresponding hyponormal operator exists, establishing a model for the operators. When the operator is subnormal, the mosaic is projection-valued [9].

The reader should be aware that restricting attention to hyponormal operators with trace class self-commutators is still a very general situation with many interesting examples. That this is the case is underlined by another of the milestones in the development of this theory. Say that an operator T is *m-multicyclic* if there are m vectors $g_1, \dots, g_m$ in $\mathscr{H}$ such that $\mathscr{H}$ is the closed linear span of $\{u(T)g_j : 1 \le j \le m$ and u is a rational function with poles off $\sigma(T)\}$ and no set of fewer than m vectors can be found to act as generators in this sense. Berger and Shaw [1, 2] showed that if T is an m-multicyclic hyponormal operator, then T has a trace class self-commutator and, moreover,

$$\operatorname{trace}[T^*, T] \le \frac{m}{\pi} \operatorname{Area}(\sigma(T)).$$

Thus the model using singular integral operators applies to all m-multicyclic hyponormal operators.

In addition to its importance in establishing the relevance of studying hyponormal operators with trace class self-commutators, the Berger-Shaw theorem has deep importance on its own. As one example of this importance, it can be used to give a simple proof of Putnam's inequality [18]: For any hyponormal operator T, $\|T^*T - TT^*\| \le (1/\pi)\operatorname{Area}(\sigma(T))$. Here is the proof. Fix a vector f with $\|f\| \le 1$ and let $\mathscr{K} \equiv$ closed linear span of $\{u(T)f : u$ a rational function with poles off $\sigma(T)\}$. If $T_1 = T|\mathscr{K}$, then T_1 is a 1-multicyclic hyponormal operator. By the Berger-Shaw theorem and the fact that $\|T_1^* f\| \le \|T^* f\|$, we get that $\langle [T^*, T]f, f\rangle = \|Tf\|^2 - \|T^*f\|^2 \le \|T_1 f\|^2 - \|T_1^* f\|^2 = \langle [T_1^*, T_1], f, f\rangle \le \operatorname{tr}[T_1^*, T_1] \le \frac{1}{\pi}\operatorname{Area}(\sigma(T_1)) \le \frac{1}{\pi}\operatorname{Area}(\sigma(T))$.

Since f was chosen arbitrarily among the unit vectors, the result follows.

Note that Putnam's inequality implies that if the spectrum of a hyponormal operator has zero area, then it must be normal.

This seems like a good place to mention another of the most significant results connected with this subject. In [5] Scott Brown proved that if T is a hyponormal operator and $C(\sigma(T)) \neq R(\sigma(T))$, the uniform closure of the rational functions with poles off $\sigma(T)$, then T has a nontrivial invariant subspace. Of course functions in $R(\sigma(T))$ are analytic on the interior of $\sigma(T)$, so any hyponormal operator whose spectrum has nonvoid interior satisfies the hypothesis of Brown's theorem. On the other hand the Hartogs-Rosenthal theorem implies that if $\sigma(T)$ has zero area, then $R(\sigma(T)) = C(\sigma(T))$; but as mentioned above, such hyponormal operators are normal and the Spectral Theorem implies that normal operators have invariant subspaces. There are, however, examples of pure hyponormal operators such that $R(\sigma(T)) = C(\sigma(T))$. So Brown's theorem does not establish the existence of invariant subspaces for the arbitrary hyponormal operator, but it comes very close and does cover all but the more pathological examples. This might be contrasted with the situation for subnormal operators, where Brown [4] has shown that every such operator has a nontrivial invariant subspace. (Also see [20] for a remarkably easy proof that can be presented in an elementary course in functional analysis.)

One of the principal tools used by Brown in his proof of the existence of invariant subspaces for hyponormal operators is a similarity model developed by one of the authors of the monograph presently under review [16]. Later both of the present authors obtained a complete unitary invariant for hyponormal operators and a consequent model. This invariant is a certain type of operator-valued distribution defined on the plane, and therein lies the rub. This invariant is very difficult to calculate for examples and certain fundamental questions about hyponormal operators cannot be answered using the model. For example, how does one distinguish the subnormal operators using the Martin-Putinar model? Can the model be used to recapture Brown's theorem? This is not a criticism of the result, but rather an indication that much work remains. Perhaps time will produce techniques for squeezing answers from this operator-valued distribution.

A similar comment can be made about the Carey–Pincus mosaic for hyponormal operators with trace class self-commutators. As stated before, the subnormal operators can be distinguished here but it is difficult to compute this invariant for some common examples.

On another front, Xia [23, 24] has produced a model for subnormal operators which he has applied to the solution of a problem concerning these operators.

So from a time when there were no real models for hyponormal operators, we have come upon a plethora of sets of complete unitary invariants. Each of these models has certain virtues, but no one has ever shown relationships between them.

It is probably the case that the present monograph is the result of the authors' work on their model. Indeed, the book gives a self-contained (except for the analysis background) treatment of their theorem, as well as the other results alluded to in this review. In fact, this book contains all the major results on hyponormal operators.

Students will, I believe, have difficulties reading this tract even if the word "student" is given its most liberal interpretation. To begin with, any aspiring reader will have to have a strong background in classical analysis. Moreover, the exposition here is somewhat raw and proofs often skip steps that are not obvious.

But let's be clear on one point. This book deserves a place on the shelf of every practicing operator theorist. In fact, it sits on my special shelf of books kept close to my desk and, as a consequence of this review, also resides in my library at home. Clearly this book defines the area in its present state.

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