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NORMAL SUBGROUPS OF SL(2, A)

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0. INTRODUCTION

We announce a characterization of the normal subgroups of $\mathrm{SL}(2, A)$ for a large class of commutative rings A including all arithmetic Dedekind domains with infinitely many units and all rings satisfying the "smallest" stable range condition (see (3.2)).

Our characterization requires several new definitions and results valid for $\mathrm{SL}(2, A)$ with A an arbitrary commutative ring. These results provide new tools for studying normal subgroups of $\mathrm{SL}(2, A)$ and should prove to be of general interest. Our main theorem extends the work of Bass, Milnor, and Serre [2] for $n \geq 3$ to the case $n = 2$ for as large a class of rings as one could reasonably expect.

If A is a commutative ring, $\mathrm{SL}(n, A)$ denotes the $n \times n$ matrices with entries in A and determinant 1. If J is an ideal in A, $L(n, A; J)$ is the normal subgroup of $\mathrm{SL}(n, A)$ consisting of those matrices congruent to scalars mod J, and $\mathrm{SL}(n, A; J)$ consists of those elements of $\mathrm{SL}(n, A)$ congruent to the identity mod J. The elementary matrix $E_{ij}(x)$, $i \neq j$, is that $n \times n$ matrix agreeing with the identity matrix except for the i, j position whose entry is x. Let $E(n, A)$ denote the subgroup of $\mathrm{SL}(n, A)$ generated by elementary matrices. If S is a subset of A, $E(n, A; S)$ is the smallest normal subgroup of $E(n, A)$ containing all elementary matrices whose nonzero off-diagonal entry comes from S.

Bass, Milnor, and Serre [2] concluded that for $n \geq 3$ and A an arithmetic Dedekind domain, N is normal in $\mathrm{SL}(n, A)$ if and only if $E(n, A; J) \subseteq N \subseteq L(n, A; J)$ for some ideal J. Furthermore, $[L(n, A; J), \mathrm{SL}(n, A)] = E(n, A; J)$. (If H and

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K are subgroups of a group G, $[H, K] = \langle\{h^{-1}k^{-1}hk | h \in H, k \in K\}\rangle$.) Actually, a similar result holds for all commutative rings when $n \geq 3$ (see [9]).

The group $L(2, A; J)$ is simply too big to provide the answer for $\mathrm{SL}(2, A)$; more carefully chosen subgroups of $L(2, A; J)$ are necessary. Given an ideal J of A, a group U of units in A/J, and an additive subgroup $\mathscr{P}$ of A closed under certain operations (a structure we have dubbed a "radix"), there is a canonically defined normal subgroup $G(J, U, \mathscr{P})$ of $\mathrm{SL}(2, A)$. These subgroups are generic in the sense that for any normal subgroup N of $\mathrm{SL}(2, A)$, the commutator $[\mathrm{SL}(2, A), N]$ is equal to the commutator $[\mathrm{SL}(2, A), G(J, U, \mathscr{P})]$, for some $J, U, \mathscr{P}$ such that $N \subseteq G(J, U, \mathscr{P})$. It follows that a subgroup N of $\mathrm{SL}(2, A)$ is normal if and only if $[\mathrm{SL}(2, A), G(J, U, \mathscr{P})] \subseteq N \subseteq G(J, U, \mathscr{P})$ for some generic group.

Sections 1 and 2 contain numerous definitions and results for arbitrary commutative rings A. The proofs are omitted in this announcement.

We include a sketch of the proof for the main theorem. Details of this work will appear in [5].

1. Radices and generic groups

(1.1) **Definition.** Let A be a commutative ring. If $X = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is any element of $\mathrm{SL}(2, A)$, set $\ell(X) = bA + cA + (a-d)A$, $u(X) = a$, and $\rho(X) = a^2 - 1 + ab$. Suppose S is a subset of $\mathrm{SL}(2, A)$. Then the level ideal of S, denoted $\ell(S)$, is the sum of the ideals $\ell(X)$ over X in S. We denote by $u(S)$ the group of units mod $\ell(S)$ generated by $u(X)$ for X in S. Finally, $\rho(S)$ is the additive group generated by $\rho(X)$ with X in S.

If N is an $E(2, A)$-normalized subgroup of $\mathrm{SL}(2, A)$, $\rho(N)$ satisfies some unusual closure properties.

(1.2) **Definition.** An additive subgroup $\mathscr{P}$ of a commutative ring A is called a *radix* provided tx^3 and $(t^3 - t)x^2 + t^2x$ are in $\mathscr{P}$ for every x in $\mathscr{P}$ and every t in A. If $\mathscr{P}$ is a radix, core $(\mathscr{P})$ is the largest ideal in $\mathscr{P}$.

(1.3) **Theorem.** *Let A be a commutative ring. If N is an $E(2, A)$-normalized subgroup of* $\mathrm{SL}(2, A)$, *then* $\rho(N)$ *is a radix.*

If A is a commutative ring, $vn_2(A)$ is the ideal $\sum_{x \in A}(x^2 - x)A$.

(1.4) **Definition.** If $\mathscr{P}$ is a radix of a commutative ring A, $G(\mathscr{P})$ is the set of all $X = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]$ in $\mathrm{SL}(2, A)$ satisfying:

(1) $\rho(X)$, $\rho(X^t)$, $\rho(X^{-1})$, $\rho(X^{-t})$ are all in $\mathscr{P}$, and
(2) $(a^2 - 1)vn_2(A) + (d^2 - 1)vn_2(A) \subseteq \mathscr{P}$.

(1.5) **Theorem.** *If $\mathscr{P}$ is a radix of a commutative ring A, then $G(\mathscr{P})$ is a normal subgroup of* $\mathrm{SL}(2, A)$. *It is the unique largest $E(2, A)$-normalized subgroup N of* $\mathrm{SL}(2, A)$ *having $\rho(N) \subseteq \mathscr{P}$. Moreover, $\rho(G(\mathscr{P})) = \mathscr{P}$.*

We are now in a position to define the generic groups we promised.

(1.6) **Definition.** Suppose A is a commutative ring, J is an ideal in A, U is a group of units mod J, and $\mathscr{P}$ is a radix. Then $G(J, U, \mathscr{P})$ is the set of all $X = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]$ in $\mathrm{SL}(2, A)$ with $\ell(X)$ contained in J, $u(X)$ in U mod J, and X in $G(\mathscr{P})$.

By (1.5), $G(J, U, \mathscr{P})$ is clearly a normal subgroup of $\mathrm{SL}(2, A)$.

(1.7) **Theorem.** *Let N be an $E(2, A)$-normalized subgroup of* $\mathrm{SL}(2, A)$. *Then*

$$G(\ell(N), u(N), \rho(N)) = \mathrm{SL}(2, A;\ \mathrm{core}(\rho(N)))E(2, A;\ \rho(N))N.$$

2. Some results on commutators

(2.1) **Definition.** If x is an element of a commutative ring A, then

$$C(x) = \begin{bmatrix} 1 & x \\ x & 1 + x^2 \end{bmatrix}.$$

If S is a subset of A, then $C(S)$ is the smallest normal subgroup of $E(2, A)$ containing $C(x)$ for each x in S.

(2.2) **Theorem.** *If A is any commutative ring and S is any subset of A, then $[E(2, A), E(2, A; S)] = C(S)$. In particular, $[E(2, A), E(2, A)] = C(A)$.*

(2.3) **Theorem.** *If A is any commutative ring and S is any subset of A, then $[E(2, A), C(S)] = C(S)$.*

(2.4) **Theorem.** *If A is any commutative ring, X is an element of* $\mathrm{SL}(2, A)$ *with a unit off the diagonal, and N is an $E(2, A)$-normalized subgroup containing X, then $C(A) \subseteq [E(2, A), N]$.*

(2.5) **Theorem.** *Let A be any commutative ring and suppose $X = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]$ is an element of* $\mathrm{SL}(2, A)$ *with a a unit in A. Let*

$x = \rho(X)$. *Then*

$$[E_{21}(1), X] = \begin{bmatrix} 1 & x \\ x & 1+x^2 \end{bmatrix}^{\left[\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]} \begin{bmatrix} a^2 & -(a-a^{-1})^2 \\ 0 & a^{-2} \end{bmatrix}.$$

If X *is in an* $E(2, A)$*-normalized subgroup* N, *then*

(i) $C(x)$ *is in* $[E(2, A), N]$,

(ii)
$$\begin{bmatrix} a^2 & -(a-a^{-1})^2 \\ 0 & a^{-2} \end{bmatrix}$$
is in $[E(2, A), N]$, *and*

(iii) $E(2, A; (a^4 - 1)A)$ *is contained in* $[E(2, A), N]$.

(2.6) **Theorem.** [10, Lemma 4.3]. *If* A *is any commutative ring and* $x, y \in A$, *then* $C(x)^{-1}C(y) = C(y-x)^{E_{12}(x)}$. *Consequently, if* N *is an* $E(2, A)$*-normalized subgroup of* $\mathrm{SL}(n, A)$, $\{x | C(x) \in N\}$ *is an additive subgroup of* A.

3. The main theorem

In this section we apply our results for arbitrary commutative rings to rings satisfying hypothesis (i), (ii), or (iii) of (3.2).

We include one preliminary result for these rings before the main theorem.

(3.1) **Theorem.** *If* A *is a commutative ring satisfying hypothesis* (i), (ii), *or* (iii) *of* (3.2) *and* J *is any ideal in* A, *then* $[\mathrm{SL}(2, A), \mathrm{SL}(2, A; J)] = C(J)$.

This theorem is an easy consequence of (2.2) and the work of Serre [7], Vaserstein [8], and Liehl [6].

(3.2) **Main Theorem.** *Suppose that*

(i) A *is a commutative* SR_2*-ring* (*see* [1, *page* 231]), *or*

(ii) A *contains infinitely many units and is a subring of the algebraic closure of the rational numbers, or*

(iii) A *contains a unit of infinite order and is a subring of the algebraic closure of* $k(x)$, *where* k *is a finite field.*

Then a subgroup N *of* $\mathrm{SL}(2, A)$ *is normal if and only if* $\rho(N)$ *is a radix and*

$$[\mathrm{SL}(2, A), G(\ell(N), u(N), \rho(N))] \subseteq N \subseteq G(\ell(N), u(N), \rho(N)).$$

Proof. We show that

$$[\mathrm{SL}(2, A), G(\ell(N), u(N), \rho(N))] = [\mathrm{SL}(2, A), N]$$

if N is normal. First apply (1.7) to get

$$G(\ell(N), u(N), \rho(N)) = \mathrm{SL}(2, A;\ \mathrm{core}(\rho(N)))E(2, A;\ \rho(N))N.$$

We need to show

$$[\mathrm{SL}(2, A), G(\ell(N), u(N), \rho(N))] \subseteq [\mathrm{SL}(2, A), N].$$

For all of the rings in question, $\mathrm{SL}(2, A) = E(2, A)$. Now (2.2) gives $[E(2, A), E(2, A;\ \rho(N))] = C(\rho(N))$, and (3.1) gives $[E(2, A), \mathrm{SL}(2, A;\ \mathrm{core}(\rho(N))] = C(\mathrm{core}(\rho(N)))$. Hence we need only show $C(\rho(N)) \subseteq N$.

Suppose J is the largest ideal with $C(J) \subseteq N$. Suppose that $\mathscr{P}$ is a radix such that $C(\mathscr{P}) \subseteq N \bmod J$. Then $C(\mathscr{P}) \subseteq N \cdot \mathrm{SL}(2, A;\ J)$. Taking commutators with $E(2, A)$ and applying (3.1) gives $C(\mathscr{P}) \subseteq [E(2, A), N] \cdot C(J) \subseteq N$. This allows us to work mod J.

Consequently, if $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is in N, we can construct, with some effort, an idempotent e in A with be a unit in Ae, and $b(1-e)$ nilpotent in $A(1-e)$. Applying (2.4) and (2.5) gives $C(a^2-1+ab)$ in N. Finally, (2.6) gives $C(\rho(N)) \subseteq N$. This completes the proof.

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