The first Bulgarian version of the book was published by the Publishing House of the Bulgarian Academy of Sciences, Sofia, in 1983. The present one is the carefully edited English translation together with several improvements. Now each chapter ends with a section on Notes, which not only provides some historical background to the subject, but also updates the material. Indeed, the list of references is considerably enlarged, and Section 2.1 on Whitney's theorem is completely rewritten, including the recent achievements of the senior author concerning the boundedness of Whitney's constants.

This book for the first time introduces into a field of recent progress in approximation theory and numerical analysis. Therefore it certainly will be of great value to those working in the broad area of error analysis. The book is well organized and (almost) self-contained. In fact, the presentation of the material is introductory, proofs are worked out in detail, and the pace is leisurely. Particularly in the applications, the authors do not present the most general results but try to emphasize the underlying principles in connection with significant examples. A list of symbols and an index round out this useful publication. In all, the book nicely surveys a substantial portion of the work of the very active Bulgarian school of approximation.

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Matrix norms and their applications, by G. R. Belitskii and Yu. I. Lyubich. Translated by A. Iacob. Oper. Theory: Adv. Appl., vol. 36, Birkhäuser, Basel, Boston, Berlin, 1988, viii+209 pp., $87.50. ISBN 0-8176-2220-9

The study of ring norms may be considered to go back to the well-known papers by Murray and von Neumann on rings of operators [4], by Gelfand on commutative Banach algebras [1], and by Gelfand and Naimark on C^*-algebras [2]. In a paper on "the metrization of matric-space" [5], von Neumann investigated the properties of ring norms constructed from gauge functions on $\mathbf{R}^n$;

this work was later carried forward by Schatten [7] who showed in particular that one gets in this way precisely the unitarily-invariant ring norms. Here a gauge function is simply a norm φ on $\mathbf{R}^n$ and the corresponding ring norm is $\|A\|_\varphi = \varphi(s_1(A), \ldots, s_n(A))$, where $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ are the eigenvalues of $(A^*A)^{1/2}$. A related source is the early work of Schatten and von Neumann [6] on norms in "cross-spaces" $X \otimes Y$ of normed linear spaces X and Y. The connection is that elements of $X \otimes X^*$ may be regarded as linear operators on X via $(x \otimes f)y = f(y)x$.

The present monograph treats some of these items, such as the analysis of unitarily-invariant norms, but the bulk of it is devoted to an exposition of results published by the authors in the 1960s. In many cases the results appear here in English for the first time. Applications are mentioned a number of times (with references), for example, to computational mathematics and mathematical genetics, but for the most part the details are not given.

The book is composed of twenty-five sections grouped into four chapters: operators in finite-dimensional spaces (32 pages), spectral properties of contractions (80 pages), operator norms (44 pages), and the order structure on the set of ring norms (44 pages). At the end there are brief (but useful) comments on the literature and a bibliography of 79 items. There is no index, which makes the book significantly more difficult to use than would otherwise be the case. This difficulty is compounded by the presence of numerous typographical errors, some of which become apparent only after considerable expenditure of effort. These are common failings, but that does not make them less objectionable or more understandable.

As the title suggests, the subject is the space of ring norms on the algebra of real or complex matrices. These are linear space norms that have in addition the ring property $\|AB\| \leq \|A\|\,\|B\|$. It might be noted that because of the finite dimensionality, a sufficiently large positive multiple of any linear space norm will be a ring norm. Of particular interest are the operator norms, induced by norms on the underlying space according to the familiar expression

$$\|A\| = \sup\{\|Ax\| : \|x\| \leq 1\},$$

in which matrices are regarded as linear operators.

Chapter 1 contains the necessary preliminaries on finite-dimensional spaces and operators. Noteworthy here is the estimate $\|A^k\|^{1/k} \leq \rho(A) + 0(\ln k/k)$ of the rate of convergence of $\|A^k\|^{1/k}$

to the spectral radius $\rho(A)$, valid for finite-dimensional normed spaces. There is an appendix on "conditioning" for continuous (not necessarily linear) maps, dealing with bounds on the loss of information in computations.

Chapter 2 is an extensive treatment of the properties of linear contractions on finite-dimensional spaces. It opens with such standard matters as characterizations of operators that are contractive or isometric in some norm, stability properties of a fixed point of a smooth (nonlinear) map in terms of the derivative at the fixed point, and the structure of semigroups of contractions. There is a substantial section devoted to the group of isometries, but it is noted that in general one does not know how to tell when a given subgroup of $\mathrm{GL}(n)$ is the isometry group in some norm. The notion of critical exponent, introduced by Marik and Ptak [3], is given an extensive development. This is the least positive integer k (if any exist) such that for all operators A, $\|A^k\| = \|A\| = 1$ implies $\rho(A) = 1$. For example, the critical exponent of the n-dimensional ℓ_∞-space is shown to be $n^2 - n + 1$. This is accomplished with the help of ideas and results from graph theory, some of which are also used to carry out the reduction (via a permutation of rows and columns) of a given matrix to "block triangular form" with the diagonal blocks in "block cyclic form." The general theory developed so far is then used to study the asymptotic behavior of powers of nonnegative matrices (i.e., those with only nonnegative entries); the results apply in particular to transition matrices of Markov chains and incidence matrices of graphs. The chapter closes with an interesting structure theorem for nonnegative projections.

The ring norms on the $n \times n$ matrices form a partially ordered set; suprema of arbitrary bounded (pointwise) subsets exist, but in general not even finite infima do. Chapters 3 and 4 develop a considerable body of information about this set, mostly due to the authors themselves. We conclude this review by describing some of the more interesting results. The operator norms turn out to be precisely the minimal elements. This is immediate from the following facts: any ring norm has an operator norm minorant (easy); if two operator norms are related, in the sense that one majorizes the other, then they are equal (difficult). One consequence is that not every unital (i.e., $\|I\| = 1$) ring norm is an operator norm: the supremum of any two (or more) distinct operator norms is never an operator norm. Moreover, a unital ring norm need not be the supremum of the operator norms it majorizes. There is an interesting characterization of those that are, as well as of the ring norms that majorize a unique operator norm. Maximal chains of ring norms have the order type of $[0, +\infty)$ and, finally, any order automorphism of the set of all ring norms is inner, in the sense that it is induced by an automorphism or antiautomorphism of the matrix algebra. The proof of the latter, which is lengthy, involves extending the automorphism to seminorms which are allowed to take the value $+\infty$. Thus a map of subalgebras, the domains of finiteness of these seminorms, is induced, from which it is possible eventually to construct the desired algebra automorphism or antiautomorphism.

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Vibration and coupling of continuous systems: asymptotic methods, by Jacqueline Sanchez Hubert and Enrique Sanchez Palencia. Springer-Verlag, Berlin, New York, 1989, 420 pp., $79.00. ISBN 3-450-19384-7

We have all been exposed at one time or another to a study of the motion of n point masses connected by linear springs and