BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 23, Number 2, October 1990

0273-0979/90 $1.00 + $.25 per page

Elementary geometry in hyperbolic space, by Werner Fenchel. De Gruyter Studies in Mathematics, vol. 11, Walter de Gruyter, Berlin, New York, 1989, xi+225 pp., $69.95. ISBN 0-89925-493-4

To obtain a helpful overview of the material in hand it is appropriate to begin with a brief discussion of the Möbius group in n-dimensions. Detailed accounts have been given from different perspectives by Ahlfors [1], Beardon [2], and Wilker [5].

Let $\Sigma = \Sigma^n = \{x \in \mathbf{R}^{n+1} : \|x\| = 1\}$ be the unit n-sphere in $\mathbf{R}^{n+1}$, $n \geq 2$. An $(n-1)$-sphere $\gamma = \gamma^{n-1}$ on Σ is the section of Σ by an n-flat containing more than one point of Σ and each such $(n-1)$-sphere determines an involution $\gamma : \Sigma \to \Sigma$ called inversion in γ. This inversion fixes the points of γ and interchanges other points in pairs which are separated by γ and have the property that any two circles of Σ, which pass through one of the points and are perpendicular to γ, meet again at the other point. The group generated by the set of all inversions of Σ is the n-dimensional Möbius group $\mathscr{M}_n$. Further properties of $\mathscr{M}_n$ can be inferred from the fact that it can also be defined as the group of bijections of Σ that preserves circles, or angles, or cross ratios, where a typical cross ratio of the four distinct points a, b, c, d belonging to Σ is the number $(\|a-b\| \, \|c-d\|)/(\|a-c\| \, \|b-d\|)$.

Let $\Pi = \Pi^n = \mathbf{R}^n \cup \{\infty\}$. Stereographic projection from Σ to Π transfers the Möbius group $\mathscr{M}_n$ to Π where it is natural to think of it as the group generated by reflections in $(n-1)$-flats and inversions in $(n-1)$-spheres. All the essential properties of the action of $\mathscr{M}_n$ are preserved in the transfer to Π because stereographic projection is induced by an inversion one dimension higher. Thus Σ and Π provide useful alternative models for viewing inversive n-space from a Euclidean perspective; to enter into the full spirit of their equivalence one need only remember that an inversive m-sphere in Π, $1 \leq m \leq n-1$, can equally well mean a Euclidean m-sphere or a Euclidean m-flat augmented by the point ∞. Since Σ^n and Π^n sit naturally in Π^{n+1}, we can perform extensions of their transformations and regard the copies of $\mathscr{M}_n$ associated with them as conjugate subgroups of $\mathscr{M}_{n+1}$. The extension of a Möbius transformation $h = \gamma_1\gamma_2 \cdots \gamma_k$ from $X^n = \Sigma^n$ or Π^n to Π^{n+1} is given by $H = \Gamma_1\Gamma_2 \cdots \Gamma_k$ where Γ_i is the inversive n-sphere orthogonal to X^n and intersecting it in $\Gamma_i \cap X^n = \gamma_i$.

The process of extension is useful in cataloguing the conjugacy classes of $\mathscr{M}_n$ [6]. A Möbius transformation $h: \Sigma \to \Sigma$ whose extension has a fixed point inside Σ is conjugate to one whose fixed point is the center of Σ. These conjugacy classes are the various (s, t)-elliptics, and by combining certain pairs of real coordinates x_1, x_2 into complex ones $z = x_1 + ix_2$, we can exhibit the representative transformation in Euclidean canonical form

$$(z_1, z_2, \cdots, z_s, x_{2s+1}, x_{2s+2}, \cdots, x_{n+1}) \to (e^{2i\theta_1} z_1, e^{2i\theta_2} z_2, \cdots, e^{2i\theta_s} z_s, (-1)^t x_{2s+1}, x_{2s+2}, \cdots, x_{n+1})$$

as the commuting product of s rotations and $t = 0$ or 1 reflections, $2s + t \leq n + 1$. A Möbius transformation $h: \Sigma \to \Sigma$ whose extension does not have a fixed point inside Σ must, by the Brouwer theorem, have a fixed point on Σ. For these transformations it is convenient to pass to Π with ∞ fixed. In Π we find either an (s, t)-hyperbolic in Euclidean canonical form

$$(z_1, z_2, \cdots, z_s, x_{2s+1}, x_{2s+2}, \cdots, x_n, x_{n+1}) \to e^{2\delta}(e^{2i\theta_1} z_1, e^{2i\theta_2} z_2, \cdots, e^{2i\theta_s} z_s, (-1)^t x_{2s+1}, x_{2s+2}, \cdots, x_n, x_{n+1})$$

as the commuting product of a dilatation, s rotations, and $t = 0$ or 1 reflections, $2s + t \leq n$; or an (s, t)-parabolic in Euclidean canonical form

$$(z_1, z_2, \cdots, z_s, x_{2s+1}, x_{2s+2}, \cdots, x_{n-1}, x_n, x_{n+1}) \to (e^{2i\theta_1} z_1, e^{2i\theta_2} z_2, \cdots, e^{2i\theta_s} z_s, (-1)^t x_{2s+1}, x_{2s+2}, \cdots, x_{n-1}, x_n + 1, x_{n+1})$$

as the commuting product of a unit translation s rotations and $t = 0$ or 1 reflections, $2s + t \leq n - 1$.

In $\mathbf{R}^n$ a rotation through 2θ is the product of reflections in two $(n-1)$-flats intersecting at a dihedral θ. Thus in Π^n a general $(1, 0)$-elliptic with parameter θ, $0 < \theta \leq \pi/2$, is the product of inversions in two inversive $(n-1)$-spheres intersecting at an angle θ. Similarly a $(0, 0)$-hyperbolic with parameter $\delta > 0$ is the product of inversions in two disjoint $(n-1)$-spheres separated by an inversive distance δ [3] and a $(0, 0)$-parabolic is the product of inversions in two tangent $(n-1)$-spheres. The canonical form of a general Möbius transformation in $\mathscr{M}_n$ indicates that it can be obtained as the product of at most $n+2$ inversions in $(n-1)$-spheres. Moreover, with a single exception occurring for the sense-reversing transformations which have $t = 1$, these $(n-1)$-spheres come in pairs which determine the conjugacy class parameters (θ's and perhaps δ or a tangency) but they are otherwise mutually orthogonal. Since inversions in orthogonal $(n-1)$-spheres commute, the first members of each pair can be grouped together, and similarly the remaining ones, to exhibit the Möbius transformation as the product of two involutions. Conversely, when two involutions in the Möbius group are multiplied together, the conjugacy class parameters of the product transformation can indicate inversively significant facts about the fixed point sets belonging to the involutions. This approach to what we might call generalized trigonometry is the underlying theme of the book under review and also its raison d'être.

The book explores this theme in connection with an introduction to hyperbolic 3-space based on the half-space model

$$U = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 > 0\} \qquad \text{with metric } dh_1 = \frac{ds}{x_3}$$

and the unit ball model

$$B = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : \|x\| < 1\} \qquad \text{with metric } dh_2 = \frac{2ds}{1 - \|x\|^2}.$$

At first sight these models seem quite different but from the point of view of inversive geometry they are really identical. To begin with, the metric spaces (U, h_1) and (B, h_2) can be seen to be isometric under the pairing induced by inversion in the sphere with center $(0, 0, -1)$ and radius $\sqrt{2}$. It follows at once that this inversion must conjugate the isometry group of one model into the isometry group of the other. Further consideration shows that much more is true: because the metrics assigned to the models are intimately related to the Möbius invariant cross ratio, the isometry groups are actually just stabilizer subgroups of $\mathscr{M}_3$. Since a Möbius transformation in $\mathscr{M}_3$ which fixes U or B is uniquely determined by the Möbius transformation in $\mathscr{M}_2$ which is its restriction to $U^b = \Pi^2$ or $B^b = \Sigma^2$, we see that the group of isometries of hyperbolic 3-space is isomorphic to $\mathscr{M}_2$.

Hyperbolic 3-space is more than just a metric space. It contains points, lines, and planes with geometric relations among them at least as interesting as those which occur in Euclidean geometry. The lines of the model (M, h), which can be defined as geodesics for the metric h, turn out to be Euclidean line segments or circular arcs perpendicular to the boundary M^b and meeting it in a pair of points known as the ends of the line; the planes or totally geodesic surfaces turn out to be subsets of Euclidean planes or spheres perpendicular to M^b and meeting it in lines or circles known as horizons. Thus although M^b does not belong to the model M, the lines of M correspond to pairs of points in M^b and the planes of M correspond to lines and circles in M^b [4]. In the case of the ball model this suggests the natural correspondence

$$h\text{-line} \to \text{end points } \{u_1, u_2\} \to \text{Euclidean segment } (u_1, u_2)$$
$$h\text{-plane} \to \text{horizon circle } c \to \text{Euclidean disc bounded by } c.$$

The projective model of hyperbolic 3-space which is implicit in the third column captures the incidence properties of (B, h_2) accurately because it is related to (B, h_2) by a homeomorphism of $\overline{B}$ that fixes each point of B^b. Geometrically this homeomorphism can be obtained by projecting $\overline{B}$ stereographically onto the hemisphere $x_4 < 0$ of the 3-sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ and then projecting this set orthogonally back onto $\overline{B}$. A great deal of hyperbolic geometry is rendered accessible by using the conformal models (U, h_1) and (B, h_2) for computation and the projective model to clarify incidence relations.

For example, the set of projective lines through a point is called an elliptic, parabolic, or hyperbolic line bundle as the point in question lies inside B^b, on B^b, or outside B^b. When an elliptic bundle is viewed in (B, h_2), it is seen to consist of all lines through a point of hyperbolic 3-space and its orthogonal trajectories are seen to be h_2-spheres with this point as center. These spheres are represented in the model by a nest of Euclidean spheres lying entirely inside the model and by referring to the special case in which the common center is the center of the model, we see that their intrinsic geometry is the usual spherical geometry of appropriate curvature. A parabolic bundle through $u \in B^b$ consists of parallel lines with common end u and its orthogonal trajectories are horospheres represented in the model by Euclidean spheres tangent to B^b at u. By passing to the model (U, h_1) with $u = \infty$, we see that the intrinsic geometry of these horospheres is just Euclidean plane geometry. Finally, a hyperbolic line bundle turns out to be the set of ultraparallel lines perpendicular to an h_2-plane with horizon circle c and its other orthogonal trajectories turn out to be equidistant surfaces lying at various distances away from this plane and represented in the model by spherical caps which meet B^b in the horizon circle c; the intrinsic geometry of these equidistant surfaces is hyperbolic plane geometry of appropriate curvature. Thus the geometry of hyperbolic 3-space includes in a natural way spherical, Euclidean, and hyperbolic plane geometry.

Fenchel unfolds the full details of this background material on hyperbolic geometry with precision and clarity that attest to a lifetime's affection for the subject. Then he goes on to develop the machinery necessary to extract metrical properties of finite sets of points, lines, and planes in hyperbolic 3-space from their embedding as point reflections, half-turns, and plane reflections in the isometry group of this space. In one application of this procedure he derives the trigonometrical relations of hyperbolic and spherical triangles by specializing appropriately the trigonometrical relations for a nonplanar right-angled hexagon. The relations for this hexagon are obtained by considering the half-turns about its edges and trace relations for the matrices in $\mathrm{SL}(2, \mathbf{C})$ which represent them. A glimpse at the method can be given in a brief discussion of half-turns.

The product of half-turns about two lines l and m is a parallel displacement ($(0, 0)$-parabolic), a rotation through 2θ about a line n($(1, 0)$-elliptic with angle θ), a translation through 2δ along n ($(0, 0)$-hyperbolic with parameter δ), or a twist equal to the commuting product of the preceding rotation and translation ($(1, 0)$-hyperbolic with parameters δ and θ) as the lines l and m are parallel or have a common perpendicular n and intersect at an angle θ, are ultraparallel and separated by a distance δ, or are skew and separated by the distance δ and angle θ. Passing to the model (U, h_1) and using the fact that its direct isometries correspond to Möbius transformations of $U^b = \Pi^2 = \mathbf{R}^2 \cup \{\infty\} = \mathbf{C} \cup \{\infty\}$ which can be written in the form $z \to (az+b)/(cz+d)$ we find that the canonical forms of the transformations above correspond to the matrices

$$\pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \pm\begin{pmatrix} e^{\delta} & 0 \\ 0 & e^{-\delta} \end{pmatrix}, \pm\begin{pmatrix} e^{\delta+i\theta} & 0 \\ 0 & e^{-\delta-i\theta} \end{pmatrix}$$

in $\mathrm{SL}(2, \mathbf{C})$ and their traces report the conjugacy class parameters in an invariant way.

To handle the inevitable ambiguity of sign in $SL(2, \mathbf{C})$, the author orients his lines so that if l has ends u and u' taken in that order it corresponds to the half-turn matrix

$$\mathbf{l} = \frac{i}{u' - u} (u + u' \quad -2uu'\partial \quad -u - u') .$$

Then trace relations for products of these matrices based on the formula

$$\operatorname{tr} \mathbf{a} \operatorname{tr} \mathbf{b} = \operatorname{tr} \mathbf{ab} + \operatorname{tr} \mathbf{a}^{-1}\mathbf{b}$$

give the desired trigonometrical relations without ambiguity of sign. Full details, including conventions to handle special position and degeneracy and additional machinery to handle opposite isometries, must await the reader's own study of this intriguing book. For a first perusal that quickly reaches the most accessible parts of the main results, I recommend §§I.3, V.3, VI.2, and VI.5 and 6.

REFERENCES

1. Lars V. Ahlfors, *Möbius transformations in several dimensions*, Ordway Professorship Lectures in Mathematics, University of Minnesota, 1981.
2. Alan F. Beardon, *The geometry of discrete groups*, Graduate Texts in Math., vol. 91, Springer-Verlag, Berlin and New York, 1983.
3. H. S. M. Coxeter, *Inversive distance*, Ann. Mat. Pura Appl. (4) **71** (1966), 73–83.
4. ——, *The inversive plane and hyperbolic space*, Abh. Math. Sem. Univ. Hamburg **29** (1966), 217–242.
5. J. B. Wilker, *Inversive geometry*, The Geometric Vein (Coxeter Festschrift), Springer-Verlag, New York, 1981, pp. 379–442.
6. ——, *Möbius transformations in dimension* n, Period. Math. Hungar. **14** (1983), 93–99.

J. B. WILKER
UNIVERSITY OF TORONTO

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 23, Number 2, October 1990

The classical groups and K-theory, by A. J. Hahn and O. T. O'Meara. Springer-Verlag, Berlin, New York, 1989, 565 pp., $119.00. ISBN 3-540-17758-2

The term "classical groups" was coined by Hermann Weyl and used in the title of his famous book [5]. It refers to the general linear group GL_n (the group of automorphisms of an n-dimensional