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J Contractive matrix functions, reproducing kernel Hilbert spaces and interpolation, by Harry Dym. American Mathematical Society, Providence, R. I., 147 pp., $22.00. ISBN-0-8218-0722-6

The primary focus of this book is a collection of interpolation problems for matrix-valued functions. The classical versions of these problems for scalar functions are associated with the names of Nevanlinna–Pick, Caratheodory–Fejer, and Nehari. The Nevanlinna–Pick problem is to find a function analytic on the unit disk mapping the disk into itself which takes prescribed values at a finite number of prescribed points in the unit disk. In the Caratheodory-Fejer problem one seeks an analytic function mapping the unit disk into the right half plane with prescribed values for the first few Taylor coefficients at the origin. In the Nehari problem one seeks a function on the unit circle with supremum norm over the unit circle at most one which has its Fourier coefficients with negative indices equal to a prescribed sequence of numbers. The work of Nevanlinna, Pick, Caratheodory, and Fejer was in the early part of this century while that of Nehari was somewhat later. The solution to all these problems splits into two parts. First, existence of a solution is equivalent to the positive semidefiniteness of a certain matrix or the contractivity of a certain operator built from the data of the problem. Secondly, when this condition holds, either the solution is unique or there are infinitely many solutions which can be described as the image of some linear fractional map applied to the set of analytic functions mapping the unit disk into its closure.

Interest in the matrix-valued versions of all these problems has become particularly intense in the last ten to fifteen years, in large part due to the advent of a new approach in engineering, called H-infinity control theory. In control theory, matrix-valued functions arise as transfer functions of physical systems. Input–output stability of the system, i.e. the requirement that bounded input signals lead to bounded output signals, leads to the requirement that the transfer function be analytic in a half plane or on the unit disk. Internal stability of a feedback configuration, i.e. the requirement that all internal signals resulting from bounded input signals be bounded, leads to additional interpolation conditions on the transfer function. Emphasis on good performance in spite of model uncertainty and in the presence of the worst expected case of outside disturbances leads to optimization problems on the transfer function in the supremum norm rather than with respect to a quadratic form, as had been more traditional in control theory. Once this change in philosophy and emphasis was introduced, it was found that a whole series of control problems can be reduced to interpolation problems of the above classical types but for matrix-valued functions.

However, it was not possible for the engineers to apply the classical results directly to their problems, even for the case where the transfer functions are complex valued. The engineering applications require a new style of mathematics where existence and uniqueness theorems alone are not enough. The theory must also be supplemented with explicit algorithms or formulas for practical computations of solutions. Moreover in the engineering applications, the original data of the problem and the presentation of the solution may be in a different form from that to which one is accustomed in classical function theory. Mathematical work which was particularly influential for engineers in the beginnings of the H-infinity theory was that of Adamian–Arov–Krein on the matrix Nehari problem and that of Sarason on the Nevanlinna–Pick and Caratheodory–Fejer problems. The work of Adamian–Arov–Krein looked particularly appealing to the engineers, since it involved a one-step extension procedure, an iterative construction where one updated the data one step at a time. Not until the incisive work of the engineer Keith Glover, who solved the problem over again from scratch in terms of a realization of the given function as a transfer function of a linear system, was there a solution of the matrix Nehari problem which was of practical use for engineering applications. The mathematician interested in an introduction to H-infinity control theory should consult [3].

There are now a number of methods for solving these matrix interpolation problems. The method of Sarason leads to the commutant lifting method; a forthcoming book [2] is entirely devoted to this topic. Another forthcoming book [1] develops an approach to interpolation problems based on a refined notion of zero and pole structure for rational matrix functions. It has also been worked out how to solve these problems iteratively by knocking off one interpolation condition at a time and using a matrix Schwarz lemma, very much in the spirit of Nevanlinna's original work. Probably the first work on an operator version of the Nevanlinna–Pick problem was that of Sz.-Nagy–Koranyi in 1958, who settled the existence question by encoding the problem into the structure of a reproducing kernel Hilbert space built from the data. To obtain the linear fractional formula for all solutions from the Sz.-Nagy–Koranyi approach, it now seems apparent that there are two ways to go. First, one can use work of Krein–Langer based on resolvent formula analysis which parametrizes all unitary extensions of a given partially defined isometric operator; this approach is being pursued by a group based at the University of Groningen (Alpay–Bruinsma–Dijksma–de Snoo). Secondly, one can use ideas of de Branges on Hilbert spaces of analytic functions to solve the problem completely in the context of reproducing kernel Hilbert spaces; this is the approach developed in the book by Harry Dym.

The main emphasis of the book is on rational matrix functions and finite-dimensional reproducing kernel Hilbert spaces. In particular one sees how a particular choice of basis for a reproducing kernel Hilbert space leads to an explicit formula for an object of interest. It is also satisfying to see how the representation theorems of de Branges on the structure of reproducing kernel Hilbert spaces lead to linear fractional representations for the set of all solutions of an interpolation problem as well as generalized representation theorems for shift-invariant subspaces of the Beurling–Lax–Halmos type. Only the simplest matrix interpolation problems are treated in complete detail; the bitangential Nevanlinna-Pick problem is treated only sketchily in the supplementary notes. But enough is here for the reader to appreciate the full potential of the method, and a more thorough, complete treatment of the topic is promised by the author in a forthcoming book. Also included in the book is introductory material on mapping properties of linear fractional tranformations; the treatment here is more algebraic than the geometric treatment via the Grassmannian as in [4].

As an application the book treats the lossless inverse scattering problem, a topic on which the author has a lot of experience with the engineer Patrick Dewilde. The terminology comes from network theory. Mathematically, the problem simply is to represent a given Schur-class function, i.e. an analytic function mapping the unit disk into itself, as a J-lossless linear fractional map composed on another Schur-class function. As this is exactly the form of representation for solutions of the interpolation problems treated in the earlier sections of the book, a solution can be found corresponding to any choice of interpolation data satisfied by the given Schur-class function. Elementary factors corresponding to solutions of interpolation conditions on the boundary have a special form called Brune sections by the engineers. Elementary factors corresponding to a single interpolation condition in the interior give rise to the more familiar Blaschke–Potapov factors.

Other topics covered in the book include matrix completion problems and a discussion of maximum extropy solutions for the whole gamut of problems discussed in the book. The matrix completion problem is a finite matrix analogue of the Caratheodory–Fejer function theory problem. In the matrix problem one is given the entries of a matrix only along a certain number of central diagonals and asked to fill in the remaining entries so the resulting matrix is positive definite. The existence criterion, due to Dym–Gohberg, is the obvious necessary condition that all determinants coming from symmetric submatrices consisting of known entries be positive. What's more, a simple inductive construction yields a linear fractional parametrization of all possible positive definite completions in terms of strictly contractive triangular matrices of an appropriate size. Choosing the maximum entropy solution of an interpolation or matrix completion problem is a way of picking out one particular solution in situations where the solution is not unique. It turns out that in most problems this solution is easily computable since it arises simply by plugging zero into the linear fractional formula which parametrizes the set of all solutions. Analytically the maximum entropy solution F is the one which maximizes an expression involving $\log\det F$. The original physical interpretation comes from statistics and information theory but recently an interpretation of risk sensitivity has been found in the context of control problems.

This book develops a particular approach to interpolation and matrix completion problems at a leisurely pace. Many of the details are sketchy; a more definitive treatment is promised later. Also there is no introduction to give the reader an orientation as to how these types of results fit into a larger framework, such as

H-infinity control. Nevertheless the book is a good introduction into an active area of research and the reader who is willing to invest some time and effort should be well rewarded.

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Operators and representation theory: Canonical models for algebras of operators arising in quantum mechanics, by P. E. T. Jorgensen. North-Holland Mathematics Studies, vol. 147, North-Holland, Amsterdam, 1988, viii+338 pp., Dfl. 175.00. ISBN 0-444-70321-7

The subtitle of the book under review is *Canonical models for algebras of operators arising in quantum mechanics*. In the preface, the author states that he has "picked certain subjects from the theory of operator algebras, and from representation theory, and showed that they may be developed starting with Lie algebras, extensions, and projective representations." The distinctive point of view arises from the consideration of algebras of *unbounded* operators, which fall outside the usual theory of C^* algebras and von Neumann algebras. Analytic properties of these operators, such as essential self-adjointness, are treated using spaces of C^∞ and analytic vectors for an appropriate Lie group action.

The basic operator of interest here is the Hamiltonian (total energy) operator H of a quantum-mechanical system. The problem is to determine the spectrum and (generalized) eigenvector decomposition of this self-adjoint operator. In Sophus Lie's creation of