BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 23, Number 2, October 1990

SHAPE OPTIMIZATION FOR DIRICHLET PROBLEMS: RELAXED SOLUTIONS AND OPTIMALITY CONDITIONS

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ABSTRACT. We study a problem of shape optimal design for an elliptic equation with Dirichlet boundary condition. We introduce a relaxed formulation of the problem which always admits a solution, and we find necessary conditions for optimality both for the relaxed and the original problem.

Let Ω be a bounded open subset of $\mathbf{R}^n (n \geq 2)$, let $f \in L^2(\Omega)$, and let $g: \Omega \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory function (i.e. $g(x, s)$ measurable in x and continuous in s) such that

$$|g(x, s)| \leq a_0(x) + b_0|s|^2 \qquad \forall (x, s) \in \Omega \times \mathbf{R},$$

for suitable $a_0 \in L^1(\Omega)$ and $b_0 \in \mathbf{R}$. We consider the following optimal design problem:

$$(1) \qquad \min_{A \in \mathscr{A}(\Omega)} \int_\Omega g(x, u_A(x))\, dx,$$

where $\mathscr{A}(\Omega)$ is the family of all open subsets of Ω, and u_A is the solution of the Dirichlet problem

$$(2) \qquad -\Delta u_A = f \text{ in } A, \qquad u_A \in H_0^1(A),$$

extended by 0 in $\Omega \backslash A$.

It is well known that, in general, the minimum problem (1) has no solution (see for instance Example 2). The reason is that, although the solutions u_{A_h} of (2) corresponding to a minimizing sequence (A_h) of (1) always admit a limit point u in the weak (not necessarily in the strong) topology of $H_0^1(\Omega)$, we can not find, in general, an open subset A of Ω such that $u = u_A$. On the contrary, it can be proved (see [4]) that the limit function u is the solution of a relaxed Dirichlet problem of the form

$$(3) \qquad -\Delta u + \mu u = f \text{ in } \Omega, \qquad u \in H_0^1(\Omega) \cap L^2(\Omega; \mu),$$

Received by the editors May 17, 1989 and, in revised form, November, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 49A50.

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0273-0979/90 \$1.00 + \$.25 per page

for a suitable nonnegative measure μ which vanishes on all sets of (harmonic) capacity 0, but may take the value $+\infty$ on some subsets of Ω. Following [3], we shall denote by $\mathscr{M}_0(\Omega)$ the class of all measures with the properties considered above.

The precise meaning of equation (3) is the following:

$$\int_\Omega DuD\varphi\,dx + \int_\Omega u\varphi\,d\mu = \int_\Omega f\varphi\,dx \tag{4}$$

for every $\forall\varphi \in H_0^1(\Omega) \cap L^2(\Omega;\mu)$, where the pointwise value of an H^1 function is defined as usual up to sets of capacity 0.

If S is a Borel subset of Ω, the measure ∞_S defined by

$$\infty_S(B) = \begin{cases} 0 & \text{if } B\cap S \text{ has capacity } 0 \\ +\infty & \text{otherwise}, \end{cases} \tag{5}$$

belongs to $\mathscr{M}_0(\Omega)$. Note that if S is closed in Ω, then problem (3) reduces to problem (2) with $A = \Omega\backslash S$ and $\mu = \infty_S$. The relaxed formulation of the optimization problem (1) is then:

$$\min_{\mu\in\mathscr{M}_0(\Omega)} \int_\Omega g(x, u_\mu(x))\,dx, \tag{6}$$

where u_μ is the unique solution of the relaxed Dirichlet problem (3) in the sense given by (4).

The following theorem follows easily from the compactness and density results for relaxed Dirichlet problems proved in [1] (Theorem 2.38) and [4] (Theorem 4.16).

Theorem 1. *Problem* (6) *admits a solution, and*

$$\min_{\mu\in\mathscr{M}_0(\Omega)} \int_\Omega g(x, u_\mu(x))\,dx = \inf_{A\in\mathscr{A}(\Omega)} \int_\Omega g(x, u_A(x))\,dx. \tag{7}$$

We now give an example where problem (1) has no solution.

Example 2. Assume that $f(x) > 0$ a.e. in Ω, let w be the solution of

$$-\Delta w = f \text{ in } \Omega, \qquad w \in H_0^1(\Omega), \tag{8}$$

and let $g(x, s) = |s - cw(x)|^2$, with $0 < c < 1$. Then the relaxed problem (6) attains its minimum value 0 at the measure μ defined by

$$\mu(B) = \frac{1-c}{c}\int_B \frac{f}{w}\,dx$$

which corresponds to $u_\mu = cw$. On the other hand, it is clear from (2) and (8) that there are no domains A for which $g(x, u_A(x)) =$

0 a.e. in Ω. By (7) this implies that the original problem (1) has no solution.

Our goal is to find optimality conditions for the solutions of problem (6). We recall that the fine topology on Ω is the weakest topology on Ω for which all superharmonic functions are continuous. For a systematic study of properties of the fine topology we refer to Doob [5], Part 1, Chapter XI. Let μ be a minimum point of (6) and let $u = u_\mu$. By $A = A(\mu)$ we denote the set of all $x \in \Omega$ having a fine neighborhood V such that $\mu(V) < +\infty$, and by μ_A the restriction of μ to A; it is clear that A is finely open in Ω. By $\partial^* A$ and $cl^* A$ we denote the fine boundary and the fine closure of A in Ω.

Proposition 3. *There exist a Radon measure $\nu \in \mathcal{M}_0(\Omega)$ carried by $\partial^* A$, and a continuous linear map $T\colon L^2(\Omega) \to L^2(\partial^* A, \nu)$ such that, if $h \in L^2(\Omega)$ and $w \in H_0^1(\Omega) \cap L^2(\Omega; \mu)$ is a solution of*

$$-\Delta w + \mu w = h \quad in\ \Omega$$

in the sense given by (4), then

$$\int_A Dw D\varphi\, dx + \int_{\partial^* A} T(h) \varphi\, d\nu + \int_A w\varphi\, d\mu_A = \int_{cl^* A} h\varphi\, dx$$

for every $\varphi \in H_0^1(\Omega)$.

If A is an open set with a smooth boundary and $\mu_A(B) = \int_{B\cap A} \vartheta\, dx$ with $\vartheta \in L^\infty(\Omega)$, an integration by parts leads to the form

$$\nu(B) = -\int_{B\cap\partial A} \frac{\partial W}{\partial n}\, d\sigma, \qquad T(h) = \frac{\partial w/\partial n}{\partial W/\partial n} \tag{9}$$

where σ denotes the surface measure on the (Euclidean) boundary ∂A of A, n is the outer unit normal to A, and W is the solution of the Dirichlet problem

$$-\Delta W = 1 \text{ in } A, \qquad W \in H_0^1(A).$$

In addition to the previous hypotheses, we assume now that $g(x, s)$ is continuously differentiable with respect to s and that

$$|g_s(x, s)| \le a_1(x) + b_1|s| \qquad \forall (x, s) \in \Omega \times \mathbf{R}$$

for suitable $a_1 \in L^1(\Omega)$ and $b_1 \in \mathbf{R}$.

In order to give our optimality conditions, we introduce the adjoint equation

$$-\Delta v + \mu v = g_s, \qquad v \in H_0^1(\Omega) \cap L^2(\Omega; \mu), \tag{10}$$

where g_s denotes the function $g_s(x, u(x))$. We denote by v the solution of (10) in the sense given by (4), with f replaced by g_s, and we set

$$\alpha = T(f), \qquad \beta = T(g_s). \tag{11}$$

Our main result is the following theorem.

Theorem 4. *Let $\mu = \infty_S + \mu_A$ be a solution of problem (6), let $u = u_\mu$ be the corresponding solution of (3), and let v be the solution of the adjoint equation (10). If $A = A(\mu)$, then $u = v = 0$ on $\Omega \backslash A$ (up to a set of capacity 0), and*

(a) *$uv \leq 0$ a.e. on A,*
(b) *$\alpha\beta \geq 0$ ν-a.e. on $\partial^* A$,*
(c) *$f(x) g_s(x, 0) \geq 0$ a.e. on $\Omega \backslash cl^* A$,*
(d) *$uv = 0$ μ_A-a.e. on A,*

where α and β are given by (11).

Suppose now that there exists an optimal domain A for the original problem (1), and that A has a smooth boundary. By (7) the measure $\mu = \infty_S$ defined by (5) with $S = \Omega \backslash A$ is a minimum point of the relaxed problem (6). Taking (9) into account, the optimality conditions of Theorem 4 become:

(a$'$) $uv \leq 0$ a.e. on A,
(b$'$) $\dfrac{\partial u}{\partial n} \dfrac{\partial v}{\partial n} \geq 0$ σ-a.e. on $\Omega \cap \partial A$,
(c$'$) $f(x) g_s(x, 0) \geq 0$ a.e. on $\Omega \backslash A$,

while condition (d) is trivial because $\mu_A = 0$. From (a$'$) and (b$'$) we obtain

$$\frac{\partial u}{\partial n} \frac{\partial v}{\partial n} = 0 \ \sigma\text{-a.e. on } \Omega \cap \partial A. \tag{b$''$}$$

The last condition is already known in shape optimization (see for instance [2, 9, 10, 13, 15]), while conditions (a$'$) and (c$'$) seem to be new. Similar relaxed formulations for different classes of optimal design problems (with Neumann or other boundary conditions) have been considered by Murat and Tartar in [8, 11, 12, 14], and by Kohn, Strang, and Vogelius in [6, 7].

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