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$L^p \to L^{p'}$ ESTIMATES FOR TIME-DEPENDENT SCHRÖDINGER OPERATORS

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MAIN RESULTS

Let $H_0 = -\Delta$, where $\Delta = (\partial/\partial x_1)^2 + \cdots + (\partial/\partial x_n)^2$ is the Laplacian in $\mathbf{R}^n$. For $t \in \mathbf{R}$, one can define $u(\cdot, t) = e^{itH_0} f$ using the spectral theorem. The function one obtains is the solution to the time-dependent Schrödinger equation

$$\begin{cases} i\partial u/\partial t + H_0 u = 0 \\ u(x, 0) = f(x). \end{cases} \tag{1}$$

Since the kernel of e^{itH_0} is $(4\pi it)^{-n/2} e^{|x-y|^2/4it}$, it is clear that the solution is dispersive in the sense that

$$\|u(\cdot, t)\|_{L^{p'}(\mathbf{R}^n)} \le Ct^{-n(1/p-1/2)} \|f\|_{L^p(\mathbf{R}^n)}, \quad t > 0, \tag{2}$$

if

$$1 \le p \le 2, \quad \text{and} \quad 1/p + 1/p' = 1. \tag{3}$$

It is well known that the local decay estimates (2) are useful in studying nonlinear Schrödinger equations (see [8, §XI.13], [11]). On the other hand little seems to be known when one replaces the free operator H_0 by more general Hamiltonians

$$H = -\Delta + V(x), \tag{4}$$

even when the potential V is in $C_0^\infty(\mathbf{R}^n)$. Obviously, one has to assume that H has no bound states for an estimate like (2) to hold for $u = e^{itH} f$. If in addition $n \ge 3$ and if one assumes that there are no half-bound states (i.e., zero resonances) the best-known decay estimates seem to be

$$\|\langle x\rangle^{-\alpha} u(\cdot, t)\|_{L^2(\mathbf{R}^n)} \le Ct^{-n/2} \|\langle x\rangle^{\alpha'} f\|_{L^2(\mathbf{R}^n)}, \quad \langle x\rangle = \sqrt{1+|x|^2}, \tag{5}$$

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if V is bounded and decays sufficiently fast at infinity, and the exponents α, α' are large enough. (see Rauch [7], Jensen and Kato [4], and Jensen [2, 3].) In the applications, the presence of the radial weights in (5) makes these estimates less useful than estimates like those in (2) (see [10]), but it is well known that they are needed if one is going to use spaces involving the square norm.

In our results, we need to assume some regularity on the potential, but we get the appropriate generalization of (2). Specifically, we shall require

$$(6)\qquad \begin{cases} \langle x\rangle^{\alpha}V(x)\colon W^{\varepsilon}\to W^{\varepsilon}, & \text{some } \alpha>2n,\ \varepsilon>0 \\ \widehat{V}\in L^{1}(\mathbf{R}^{n}). \end{cases}$$

The first assumption means that multiplication by $\langle x\rangle^{\alpha}V$ sends the usual L^2 Sobolev space W^{ε} into itself, and it would always be fulfilled when $\langle x\rangle^{\alpha}V\in \mathrm{Lip}(\varepsilon)$ (uniformly). Undoubtedly the assumption regarding α can be improved, and at the expense of simplicity we can show that $\alpha>n$ is sufficient.

If P_c denotes the projection onto the continuous part of the spectrum of H (i.e., $(0,\infty)$), then our main result is the following:

Theorem. *Let $n\geq 3$ and V satisfy* (6). *Then, if* 0 *is neither an eigenvalue nor a resonance for H,*

$$(7)\qquad \|e^{itH}P_c f\|_{L^{p'}(\mathbf{R}^n)}\leq Ct^{-n(1/p-1/2)}\|f\|_{L^{p}(\mathbf{R}^n)},\quad t>0,$$

if p and p' are as in (3).

The spectral assumptions on H are probably necessary to get the optimal decay rate as in (5) (cf. [2, 3]). But in certain cases we can show that there is slower decay as $t\to\infty$ if the assumptions are relaxed. We should also point out that when the dimension is ≥ 5 the resonance condition is automatically fulfilled since there can be no half-bound states in this case. This assertion just follows from the fact that the fundamental solution for H_0 is in L^2 near infinity when $n\geq 5$.

The proof of (7) splits into two parts: a "high-energy" estimate and a "low-energy" estimate. To be more specific, suppose that $\int E'(\lambda)\,d\lambda$ is the resolution of the identity associated to H. Fix $\beta\in C^{\infty}(\mathbf{R})$ satisfying $\beta(\lambda)=0$ for $-\infty<\lambda<1/2$ and $\beta(\lambda)=1$ for $1<\lambda<\infty$. We then define high- and low-energy functions associated to $P_c f$ by

$$f_M = \int \beta(\lambda/M) E'(\lambda) f \, d\lambda$$

$$\tilde{f}_M = \int [1 - \beta(\lambda/M)] E'(\lambda) P_c f \, d\lambda,$$

where $M > 0$ is a large number that depends on V. Note that $f_M + \tilde{f}_M = P_c f$.

The first step in the proof of (7) is to show that

$$\|e^{itH} f_M\|_{L^{p'}} \leq C t^{-n(1/p - 1/2)} \|f\|_{L^p}, \tag{7.1}$$

if M is large enough. Here it turns out that one only has to assume (6); that is, (7.1) is true for any Hamiltonian H, even if 0 is an eigenvalue or resonance and the quantity M depends only on the constants in (6). The proof of this high-energy estimate involves a bootstrapping argument which uses DuHamel's formula and two main estimates. The first one is that

$$\left\| e^{itH_0} \left(\prod_{2 \leq \nu \leq k} e^{-is_\nu H_0} V e^{is_\nu H_0} \right) e^{-is_1 H_0} V e^{is_1 H} \right\|_{(L^1, L^\infty)} \leq t^{-n/2} \|\widehat{V}\|_{L^1}^k e^{|s_1| \|\widehat{V}\|_1},$$

where $\|\cdot\|_{(L^1, L^\infty)}$ denotes the $L^1 \to L^\infty$ operator norm. This estimate follows from an argument which uses the Fourier transform. It is here that the second assumption in (6) is used. The other main ingredient in the proof of the high-energy estimate is the "local smoothing" property of the free Schrödinger equation:

$$\|\langle x \rangle^{-1/2-\varepsilon} (I + H_0)^{1/4} e^{itH_0} f\|_{L^2(\mathbf{R}^n \times [0,1])} \leq C_\varepsilon \|f\|_{L^2(\mathbf{R}^n)}, \quad \varepsilon > 0.$$

This was proved independently by Sjölin [9] and Vega [13] and Constantin and Saut [1]. In the part of the argument which uses this estimate the first assumption in (6) is needed. Here it turns out that we also have to use the fact that the operators $\beta(H/M)$ (defined by the spectral theorem) are weakly pseudolocal in the sense that their kernels multiplied by $|x - y|^N$ are integrable away from the diagonal for any N.

In the case of odd dimensions, $V \in C^\infty$ and $D^\alpha V$ decaying exponentially for all α, A. Mellin [6] has shown that certain intertwining operators for e^{itH} exist and are unitary when acting on functions whose spectrum is nonzero only for large λ. For the

Schrödinger operators associated to such potentials, his results can be used to give a more direct proof of (7.1).

The other part of the proof of (7) is establishing the low-energy estimate

$$\|e^{itH}\tilde{f}_M\|_{L^{p'}} \le Ct^{-n(1/p-1/2)}\|f\|_{L^p}. \tag{7.2}$$

Here the spectral assumption on H of course is used. The proof of this low-energy estimate is based on writing the operator involved using the spectral theorem and then using the identity

$$E'(\lambda) = \pi^1 \operatorname{Im}\{(I + R_0(\lambda + i0)V)^{-1}R_0(\lambda + i0)\}.$$

Here $R_0(\zeta) = (H_0 - \zeta)^{-1}$ denotes the free resolvent, and $R_0(\lambda + i0) = \lim_{\varepsilon\to 0_+} R_0(\lambda + i\varepsilon)$. The properties of the free resolvent are well known and one can consequently obtain the desired estimates for $E'(\lambda)$ by using results of Jensen and Kato [2–4] concerning the behavior of $(I + R_0(\lambda + i0)V)^{-1}$. If the dimension n is odd, it turns out that

$$(I + R_0(\zeta)V)^{-1}[I - \beta(H/M)] = \sum_{j=0}^{l}(i\zeta^{1/2})^j B_j + o(\zeta^{1/2}), \qquad \text{as } \zeta \in \mathbf{C}_+ \to 0,$$

in the $L^2(\langle x\rangle^{-\alpha}\, dx) \to L^2(\langle x\rangle^{-\alpha}\, dx)$ topology for certain α. The operators B_j are real and in principle can be computed explicitly for any j, using the Neumann series for $(I + R_0(\zeta)V)^{-1}$. In particular, one can show the important fact that $B_j = 0$ for odd $j < n - 2$. Using this asymptotic expansion together with certain stationary phase estimates such as

$$\int_0^\infty e^{i\lambda t}\rho(\lambda/M)(\lambda^{1/2})^j \operatorname{Im} R_0(\lambda + i0; x, y)\, d\lambda = O(t^{-n/2}), \qquad j \ge 0 \text{ even},$$

$$\int_0^\infty e^{i\lambda t}\rho(\lambda/M)(\lambda^{1/2})^j \operatorname{Re} R_0(\lambda + i0; x, y)\, d\lambda = O(t^{-n/2}[1 + |x-y|^{-(n-2)}]), \qquad j \ge n-2 \text{ odd},$$

one gets (7.2). Here $R_0(\lambda+i0; x, y)$ denotes the kernel of $R_0(\lambda+i0)$ and $\rho = 1 - \beta$, where β is as above. The argument for even dimensions is similar, except both the asymptotic expansion for $(I + R_0(\zeta)V)^{-1}$ and the stationary phase estimates that are required are more technical since the free resolvent has a more complicated form in even dimensions.

SOME APPLICATIONS

One of our main applications is an extension of a "global decay estimate" of Strichartz [12] for the free operator to include Schrödinger operators having no bound states.

Corollary. *Let $u(x, t)$ be the solution to the Schrödinger equation* (1) *where H_0 is replaced by $H = -\Delta + V$ with V satisfying* (6). *Assume that H has no bound states or half-bound states and that $n \geq 3$. Then if $f \in L^2(\mathbf{R}^n)$, it follows that u is globally in $L^{q'}$ if $q' = 2(n+2)/n$:*

$$\|u\|_{L^{q'}(\mathbf{R}^n \times \mathbf{R})} \leq C\|f\|_{L^2(\mathbf{R}^n)}. \tag{8}$$

The same conclusion holds if 0 *is neither an eigenvalue nor resonance and we assume that f is orthogonal to the bound states.*

In the free case this inequality is equivalent, by duality, to the following restriction theorem for the Fourier transform:

$$\left(\int_{\mathbf{R}^n} |\hat{g}(\xi, |\xi|^2)|^2 \, d\xi\right)^{1/2} \leq C\|g\|_{L^q(\mathbf{R}^n \times \mathbf{R})}, \quad q = 2(n+2)/(n+4).$$

To prove the global decay estimate one uses duality and then repeats the first part of the proof of this restriction theorem to see that (8) is equivalent to the inequality

$$\left\|\int_{-\infty}^{\infty} e^{i(t-s)H} g(\cdot, s)\, ds\right\|_{L^{q'}(\mathbf{R}^n \times \mathbf{R})} \leq C\|g\|_{L^q(\mathbf{R}^n \times \mathbf{R})}. \tag{8$'$}$$

Finally, the fact that optimal local decay estimates can be used to prove inequalities like (8$'$) is well known, and by using arguments similar to those in Kenig and Sogge [5] one sees that (2) yields (8$'$).

Using this global decay estimate we can extend results of Strauss [11] concerning scattering theory for nonlinear Schrödinger equations of the form

$$i\partial u/\partial t + Hu + b(|u|) \arg u = 0 \tag{9}$$

involving H as in the corollary and $b \in C^1$ satisfying

$$|b'(s)| \leq C|s|^{p-1}, \qquad \text{some } 1 + 4/n \leq p < 1 + 4/(n-2).$$

For instance, by using results in [11] together with (8) one can show that, given initial data $f_0 \in W^1$ with $\|f_0\|_{W^1} \leq \delta$ and $\delta > 0$

small, there is a unique solution u to the nonlinear initial value problem for (9) and unique $f_+, f_- \in W^1$ so that

$$\|u(\cdot, t) - e^{itH_0} f_{\pm}\|_{W^1} \to 0, \qquad \text{as } t \to \pm\infty.$$

Detailed proofs and applications concerning scattering theory for nonlinear multichannel problems will appear later.

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