BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 23, Number 2, October 1990

COMPLETE NONCOMPACT KÄHLER MANIFOLDS WITH POSITIVE HOLOMORPHIC BISECTIONAL CURVATURE

WAN-XIONG SHI

In the theory of complex geometry and Kähler manifolds, one of the famous problems is the following conjecture:

Conjecture. Suppose M is a complete noncompact Kähler manifold with positive holomorphic bisectional curvature. Then M is biholomorphic to $\mathbf{C}^n$.

Several results concerning this conjecture were obtained in the past few years. In 1981 N. Mok, Y. T. Siu, and S. T. Yau [4] proved the following result:

Theorem 1. *Suppose M is a complete noncompact Kähler manifold of complex dimension $n \geq 2$. Suppose M is a Stein manifold and the holomorphic bisectional curvature is nonnegative. Suppose M satisfies the following assumptions:*

(i) $\mathrm{Vol}(B(x_0, r)) \geq c_0 r^{2n}$, $0 \leq r < +\infty$,

(ii) $0 \leq R(x) \leq c_1 / r(x, x_0)^{2+\varepsilon}$, $x \in M$,

where $0 < c_0, c_1, \varepsilon < +\infty$ are some constants, $B(x_0, r)$ denotes the geodesic ball of radius r and centered at x_0, $\mathrm{Vol}(B(x_0, r))$ denotes the volume of $B(x_0, r)$, $r(x_0, x)$ denotes the distance between x_0 and x, and $R(x)$ denotes the scalar curvature at $x \in M$. Then M is isometrically biholomorphic to $\mathbf{C}^n$ with the flat metric.

Their result was improved by N. Mok [5] in 1984. In [5] N. Mok obtained the following result:

Theorem 2. *Suppose M is a complete noncompact n-dimensional Kähler manifold with positive holomorphic bisectional curvature and satisfies the following assumptions:*

(i) $\mathrm{Vol}(B(x_0, r)) \geq c_0 r^{2n}$, $0 \leq r < +\infty$,

(ii) $0 < R(x) \leq c_1 / r(x_0, x)^2$, $x \in M$,

Received by the editors October 24, 1989 and, in revised form, March 14, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53C55, 58G11.

where $0 < c_0, c_1 < +\infty$ *are some constants. Then* M *is biholomorphic to an affine algebraic variety.*

In this note we announce the following result:

Theorem 3. *Suppose* M *is a complete noncompact Kähler manifold of complex dimension* n *with bounded and positive holomorphic bisectional curvature. Suppose* M *satisfies the following assumptions*:

(i) $\mathrm{Vol}(B(x_0, r)) \geq c_0 r^{2n}$, $0 \leq r < +\infty$,

(ii) $\int_{B(x_0, r)} R(x)\, dx \leq c_1 r^{2n-2}$, $x_0 \in M$, $0 \leq r < +\infty$,

where $0 < c_0, c_1 < +\infty$ *are some constants. Then* M *is biholomorphic to* $\mathbf{C}^n$.

The method we used to prove Theorem 3 is to study the following Ricci flow evolution equation on Riemannian manifolds:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \tag{1}$$

where g_{ij} is the Riemannian metric on the manifold M, R_{ij} denotes its Ricci curvature tensor. Evolution equation (1) was originally introduced by R. S. Hamilton [1] in 1982. In that paper Hamilton used (1) to classify all of the three-dimensional compact Riemannian manifolds with positive Ricci curvature. Since then many works dealing with or related to equation (1) have been published; for example, one can see [2, 3, and 6].

In the case where M is a complex Kähler manifold, evolution equation (1) is strongly related to complex Monge-Ampère equation. On Kähler manifolds Monge-Ampère equation is a nonlinear elliptic type equation and (1) is the corresponding parabolic equation of Monge-Ampère equation. Under this point of view, H. D. Cao generalized the arguments established by S. T. Yau for Monge-Ampère equation in the proof of Calabi's conjecture to the corresponding Ricci flow equation and obtained a different proof of Calabi's conjecture in 1985.

Using evolution equation (1) to deform the metric g_{ij} on Riemannian manifold M, the first thing one has to consider is the short time existence for the solution of (1). In the case where M is compact, evolution equation (1) always has a smooth solution for at least a short time interval. This fact was proved by R. S. Hamilton in [1]. In the case where M is complete and noncompact, the similar statement in general is not true. One can easily find a counterexample when the curvature tensor is unbounded on the whole manifold. If we assume that M is a complete noncompact Riemannian manifold with bounded curvature tensor, then we have the following short time existence theorem which was proved in [7]:

Theorem 4. *Let* $(M, g_{ij}(x))$ *be an* n*-dimensional complete noncompact Riemannian manifold with its Riemannian curvature tensor* $\{R_{ijkl}\}$ *satisfying*

$$|R_{ijkl}(x)|^2 \leq k_0, \qquad \forall x \in M, \tag{2}$$

where $0 < k_0 < +\infty$ *is a constant. Then there exists a constant* $T(n, k_0) > 0$ *depending only on* n *and* k_0 *such that the evolution equation*

$$\begin{cases} \frac{\partial}{\partial t} g_{ij}(x, t) = -2R_{ij}(x, t) \\ g_{ij}(x, 0) = g_{ij}(x) \end{cases} \tag{3}$$

has a smooth solution $g_{ij}(x, t) > 0$ *for a short time* $0 \leq t \leq T(n, k_0)$*, and satisfies the following estimates: For any integers* $m \geq 0$*, there exist constants* $c_m > 0$ *depending only on* n, m*, and* k_0 *such that*

$$\sup_{x \in M} |\nabla^m R_{ijkl}(x, t)|^2 \leq c_m / t^m, \qquad 0 \leq t \leq T(n, k_0), \tag{4}$$

where $\nabla^m R_{ijkl}$ *denote the* m*th order covariant derivatives of* R_{ijkl}.

Sketch of the proof of Theorem 3. Now we suppose M is a complex n-dimensional complete noncompact Kähler manifold which satisfies the hypothesis in Theorem 3. Since the holomorphic bisectional curvature on M is positive and bounded, we know that the curvature tensor $\{R_{ijkl}\}$ of M satisfies condition (2) in Theorem 4 with some constant $k_0 < +\infty$. Thus from Theorem 4, we know that evolution equation (1) has a solution $g_{ij}(x, t) > 0$ on M for a short time interval.

In the next step, we want to show that the solution $g_{ij}(x, t)$ of evolution equation (1) actually exists for all time $0 \leq t < +\infty$. To prove the long time existence for the solution of (1), what we need to prove is that the curvature of $g_{ij}(x, t)$ would not tend to infinity on any finite time interval. This kind of fact and the techniques used to prove it were originally established by author in [8] under some more restricted curvature pinching and decay assumptions. Modifying and generalizing the arguments developed in [8], we can show that on any finite time interval $0 \le t < T$, the curvature of the solution $g_{ij}(x, t)$ is bounded uniformly on $M \times [0, T)$ under the hypotheses of Theorem 3. Thus we know that the solution $g_{ij}(x, t)$ of (1) actually exists for all time $0 \le t < +\infty$ under the hypotheses of Theorem 3.

Suppose $g_{ij}(x, t) > 0$ defined on $M \times [0, \infty)$ is the solution of evolution equation (1) for all time $0 \le t < +\infty$. By the assumption in Theorem 3, we know that $g_{ij}(x, t)$ is Kähler metric at time $t = 0$. Using the maximum principles for the solutions of heat equations on complete noncompact Riemannian manifolds which were developed in [8], it is easy to show that $g_{ij}(x, t)$ are Kähler metrics for all $0 \le t < +\infty$. Similar to what we did in [8], we can show that the curvature $\{R_{ijkl}(x, t)\}$ of $g_{ij}(x, t)$ and its covariant derivatives tend to zero as time $t \to +\infty$. Thus finally we obtained a flat Kähler metric on M, hence we know that M is isometrically biholomorphic to $\mathbf{C}^n$ with the new metric. The proof of Theorem 3 is complete.

The details of the proof of Theorem 3 will appear elsewhere.

References

1. R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), 255–306.
2. ——, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24** (1986), 153–179.
3. G. Huisken, *Ricci deformation of the metric on a Riemannian manifold*, J. Differential Geom. **21** (1985), 47–62.
4. N. Mok, Y. T. Siu, and S. T. Yau, *The Poincaré-Lelong equation on complete Kähler manifolds*, Compositio Math. **44** (1981), 183–218.
5. N. Mok, *An embedding theorem of complete Kähler manifolds of positive bisectional curvature onto affine algebraic varieties*, Bull. Soc. Math. France **112** (1984), 197–258.
6. W. X. Shi, *Complete noncompact three-manifolds with nonnegative Ricci curvature*, J. Differential Geom. **29** (1989), 353–360.
7. ——, *Deforming the metric on complete Riemannian manifolds*, J. Differential Geom. **30** (1989), 223–301.
8. ——, *Ricci deformation of the metric on complete noncompact Riemannian manifolds*, J. Differential Geom. **30** (1989), 303–394.

Department of Mathematics, Harvard University, Cambridge, Massachusetts

Current address: Department of Mathematics, Purdue University, West Lafayette, Indiana 47907