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Existentially closed groups, by Graham Higman and Elizabeth Scott. London Mathematical Society Monographs, New Series, vol. 3., Clarenden Press, Oxford, 1988, xiv + 156 pp., $49.95. ISBN 0-19-853543-0

SUMMARY

This volume grew out of lectures given by Higman at Oxford in 1983 and 1984 as recorded and amended by Scott. It is not a comprehensive work on e.c. groups but rather contains an ample selection of topics written at an easily accessible graduate level. Both algebraic and model-theoretic aspects of e.c. groups are highlighted. Thus, Chapter 2 gives two very diferent group-theoretic proofs that the normalizer of a finite characteristically simple subgroup of an e.c. group G is a maximal subgroup of G, as well as related results, and has considrable technical interest. [For extensions of one of these methods, see the reviewer's "A.c. groups: Extensions, maximal subgroups, and automorphisms," Trans. Amer. Math Soc. 290, (1985), 457–481.] This book contains all the results of Hickin and Macintyre's "A.c. groups: Embeddings and centralizers" (in *Word Problems* II, North-Holland, 1980) with the exception of the spectrum problem in power ω_1. After some preliminaries, Chapters 5 and 6 develop some algebraic applications of the Higman embedding theorem and its generalized version (which is deduced in the text). In particular the embedding of wreath products into e.c. groups is discussed. The highlights of these sections are

(1) a slick recursion-theoretic proof that the class of all recursively presented groups forms a minimal skeleton for an e.c. group, and
(2) the very nice theorem (due to Higman) that if $A \neq 1$ then there is no universal finitely presented group over A (which has the corollary that no e.c. group is embedded in a finitely generated subgroup of itself).

Chapter 7 develops some forcing techniques (in a game-theoretic manner) used to construct countable e.c. groups. These are employed to

(1) produce 2^{ω} countable e.c. groups with pairwise disjoint skeletons (i.e. except for subgroups with solvable word problems!) and
(2) produce an e.c. group M and a finitely generated, recursively presented subgroup $G \subset M$ such that for all $H \neq 1$, the free product $G * H$ is not embeddable in M.

The first of these results can be proved by the simple forcing argument used in Lyndon and Schupp, *Combinatorial Group Theory*, Prentice Hall, 1977, pp. 232–233. [Ziegler's classification (M. Ziegler, "Algebraisch abgeschlossene gruppen" in *Word Problems* II, North-Holland, 1980) is beyond the scope of these chapters, although Ziegler-reducibility is discussed and the result describing those groups forced into an e.c. group by the inclusion of a fixed group is given.] The final chapter deals with the first-order theory of e.c. groups, the highlight being the theorem that the embeddability of any given arithmetically related finitely generated group in an e.c. group is controlled by a first-order sentence in the e.c. group. Some results on generic groups are also given and several difficult exercises occur in the text.

Discussion

The subject of existentially closed groups currently occupies a special place in the model theory and logic of algebraic systems. This is due to a number of convergent factors, not the least of which is that this subject performs a type of wedding between recursion theory (the foundation of mathematical logic) on the one hand and group theory (the foundation of the analysis of mathematical structure) on the other. In another sense, it and related topics in other systems provide a testing ground for the classificatory concepts of model theory. E.c. groups provide an exemplary case in which powerful techniques are available to explore the details and potentials of logical complexity in a class of systems well known for their intrinsic beauty.

An *existentially closed* structure E for a class of algebraic systems Σ is a sort of "universal structure for Σ-relations" in the sense that $E \in \Sigma$ and every *finite* set of equalities and inequalities (that is, negations of equalities) of terms (or more generally of atomic formulas and their negations), containing finitely many variables as well as constants from E, which can be solved in some Σ-superstructure of E can also be solved in E. The related concept of an *algebraically closed* Σ-structure is obtained by allowing only equalities (or atomic formulas) in the finite sets of consistent conditions which must be solvable in E. As for fields (in which a.c. and e.c. objects coincide) every nontrivial a.c. group is e.c.[1]; but for the class of locally finite p-groups there are two nontrivial countable a.c. structures, one of which is the unique countable e.c. structure [Leinen and Phillips, A.c. groups in locally finite group classes, Lecture Notes in Math 1281, Group Theory Proc. Brixen/Bressanone, 1986, 85–102].

An e.c. structure is a type of *saturated structure.* (These play a crucial role in the study of nonstandard models, model-extensions, and other aspects of model theory). In simple (and favorable) terms an e.c. structure for Σ is a Σ-structure which has all finite consistent Σ-conditions built into itself and so is a sort of universe in which all Σ-algebra can be consistently performed. This is a weak type of saturation, which generally requires certain infinite sets of consistent conditions (describing element types) to be reflected. In extremely favorable circumstances (which include all locally finite classes) the e.c. Σ-structures coincide with the stronger notion of *basically saturated* structures $B \in \Sigma$ defined by an amalgamation condition: if $C \subseteq D$ are finitely generated Σ-structures with $C \subseteq B$ and D can be amalgamated with B over C in *some* Σ-structure, then there is an embedding of D into B over C. (This type of structure has been incisively studied in [B. Maier, On countable locally described structures, Ann. Pure Appl. Logic **35** (1987), 205–246].)

[1]This is because if a and $b \neq 1$ are elements of any group G and x, y are variables then the equation $b = x^{-1}axy^{-1}ay$ is consistent over G if $a \neq 1$. Thus equalities can be used to force inequalities.

Countable e.c. structures exist in almost all classes of any interest by an easy direct limit argument, whereas there are no countable basically saturated groups since there are $2^{\aleph_0}$ finitely generated groups. (This same fact implies that $2^{\aleph_0}$ countable e.c. groups exist.) Every e.c. group is simple. A countable e.c. group E is determined uniquely by its finitely generated subgroups and every isomorphism between two of these is induced by conjugation in E. Despite this homogeneity there is an incredible variety of e.c. groups. The powerful technique needed to explore the world of an e.c. group is the embedding theorem of Graham Higman which states that every group with a recursively enumerable presentation $G = \langle x_n (n \geq 1) | R \rangle$ (R recursively enumerable) is contained in a finitely presented group $H = \langle h_1, \dots, h_k | S \rangle$ (S finite) so that the $\{x_n\}$ equal a recursive set of words on $\{h_1, \dots, h_k\}$. This leads to the embedding of parts of recursion theory into group theory as well as the recursion-theoretic character of e.c. groups. From this and a simple group-theoretic construction it follows that an e.c. group G has elements satisfying any recursively enumerable, consistent (i.e. solvable in some larger group) set of implication $\rho_1 \to \rho_2 \wedge \cdots \wedge \rho_k$ or $\neg\rho_1 \to (\neg\rho_2) \wedge \cdots \wedge (\neg\rho_k)$ where the ρ_j are equations among terms containing constants from a finitely generated subgroup of G and variables $\{x_1, \dots, x_n, \dots\}$. This implies, for example, that every group with a solvable word problem is a subgroup of every e.c. group since we can effectively enumerate all relations as well as all nonrelations in such a group. It also implies that an e.c. group must be somewhat complicated: no e.c. group can possess a recursively enumerable presentation (on an infinite set of generators).[2]

The simplest e.c. group is the *homogeneous* group which is a direct limit of finitely presented groups and contains every such group. This group is arithmetically definable (at a very low level of this heirarchy) and so has a strong constructive character—the ineffectiveness involved in it is, in practical terms, no worse than our inability to perform computations among algebraic numbers (even though such computations are theoretically possible). Higman's theorem (and its relativization) allows us to show that many group theoretic constructions can be performed within any e.c.

[2]If so, then, because it is simple, it would have a solvable word problem and this would contradict the lemma below.

group. For example, if A and B are finitely generated subgroups of the e.c. group G and $A]B$ is any semidirect product of A by B, then $A]B$ can be embedded into G—we need finitely many equations to say how (a copy of) B acts on (a copy of) A and the infinite set of implications $ab = 1 \rightarrow a = 1 \wedge b = 1$ $(a \in A, b \in B)$. Even the restricted wreath product $AwrB$ can be embedded in G provided B has a solvable word problem. However, it is not necessary that the free product $A * B$ embed in G even if $|A| = 2$ and B is recursively presented, and it is not possible to perform strong analgamations in all e.c. groups. This points-up the enormous difficulty of classifying the possible collections of finitely generated groups which can comprise an e.c. group—a question which was successfully answered by Martin Ziegler ten years ago [op. cit.] by giving a high-powered recursion-theoretic classification of them. Despite this classification much work has gone into studying special types of e.c. groups such as the generic groups[3] which arise from the use of forcing to construct countable e.c. groups, as well as the first-order theories of e.c. groups and the power of first-order sentences to determine the structure of an e.c. group (which is very considerable). Work has also been done (especially by the reviewer) in studying the global properties of e.c. groups—maximal subgroups, centralizers, automorphisms, extensions, permutation representations, etc. [K. Hickin, op. cit. and Some applications of tree-limits to groups, Trans. Amer. Math. Soc. **305**, 797–839 (1988)].[4] The book under review presents a fair sampling of these endeavors and contains much of interest both to group theorists and logicians.

As an illustration of the nice fit of recursion theory with the group theory of e.c. groups we will give a short proof (after the method of Lyndon–Schupp, op. cit.) of the following—which is

[3]The concept of a generic structure, a logically technical one, can be nicely illustrated by its algebraic analog for basically saturated structures B : B is basic-generic $\Leftrightarrow$ whenever the amalgam $D \cup B$ of the previous definition is Σ-inconsistent (i.e. it cannot be embedded in a larger Σ-structure) then $D \cup B_0$ is also Σ-inconsistent for some finitely generated $B_0 \subseteq B$.

[4]A good example of a global result is the following which the reviewer [op. cit.] was able to obtain using Ziegler's "homogeneous limit" [M. Ziegler and S. Shelah, A.c. groups of large cardinality, J. Symbolic Logic **44** (1979), 130–140]: Suppose $A \subseteq B$ are e.c. groups and we can recursively enumerate generators of A as words involving finitely many elements of B. Let $\overline{A}$ be any e.c. group having the same finitely generated subgroups as A. Then $\overline{A} \subseteq \overline{B}$, where $\overline{B}$ is an e.c. group having the same finitely generated subgroups as B.

the main result in Chapter 7 of the book under review:

Theorem. *There exist* $2^{\aleph_0}$ *nonisomorphic countable e.c. groups no two of which have isomorphic finitely generated subgroups with unsolvable word problems.*

The proof consists of a recursion-theoretic lemma and then a simple "forcing" construction.

Lemma. *Every e.c. group has a finitely generated subgroup with an unsolvable word problem.*

The proof of this lemma uses *recursively inseparable* sets of natural numbers—a consequence of elementary recursion theory [H. Rogers, Theory of Recursive Functions and Effective Computability, p. 94]—and Higman's embedding theorem.[5]

Proof of the Lemma. Let I, J be recursively inseparable sets of natural numbers. (This means that I, J are recursively enumerable: $I \cap J = \varnothing$; and $I \subseteq X$, $X \cap J = \varnothing$ implies X is not recursive—these can be obtained as $I = \{n | \varphi_n(n) = 0\}$ and $J = \{n | \varphi_n(n) = 1\}$ where φ_n is the nth partial recursive function under some Gödel numbering.) Now consider the recursively enumerable presentation for a group P with generators $\{y_k, t_k | k \geq 1\} \cup \{z\}$ and relations $\{y_k = 1 | k \in I\} \cup \{z = t_k^{-1} y_k t_k | k \in J\}$. Using Higman's theorem we embed P in a group $H = \langle q_1, \ldots, q_n | R \rangle$ (R finite). If E is an e.c. group we choose $w_1, \ldots, w_n \in E$ so that the map $q_1 \to w_1, \ldots, q_n \to w_n$ defines a homomorphism φ from H into E (that is, the relations R are satisfied by $w_1, \ldots, w_n$) and such that $\varphi(z) \neq 1$. Thus $I \subseteq X = \{i | \varphi(y_i) = 1\}$ and $X \cap J = \varnothing$ because $\varphi(y_k) = 1$ and $k \in J$ implies (via the relations of $P \subseteq H$) that $\varphi(z) = 1$. Hence X is not recursive, proving that $\varphi(H) \subseteq E$ has an unsolvable word problem.

The Construction

Let $X = \{x_n | n \geq 1\}$ be variables and F be the free group on X. We will construct presentations $\langle X | R \rangle$ of e.c. groups $G = F/R^F$ by specifying increasing finite sets of relations $R_n \subseteq F$ so that $R = \bigcup\{R_n | n \geq 1\}$. At the same time we will specify

[5]With a bit more work the construction used in this proof can be used to obtain a finitely presented group G such that no nontrivial image of G has a solvable word problem.

increasing finite sets $I_n \subseteq F$ whose elements will remain *nontrivial* in G. The requirements needed to guarantee that G is an e.c. group are (i) $I_1 = \{x_1\}$ will be a nontrivial element of G, and, letting $\mathscr{W} = \{W_k | k \geq 1\}$ be all finite sets of words involving the $\{x_n | n \geq 1\}$ as well as new variables $T = \{t_n | n \geq 1\}$ (and their inverses), (ii) For each $W \in \mathscr{W}$ there is a step n of our construction such that (a) if the relations W are consistent with R_n (that is, $\langle X | R_n \rangle$ is a subgroup of $\langle X \cup T | R_n \cup W \rangle$) we choose variables from X not occurring in R_n and put $R_{n+1} = R_n \cup W'$, where W' results from substituting the new X-variables for all T-variables occurring in W; and (b) if W is inconsistent with R_n then we put $I_{n+1} = I_n \cup \{z\}$ where $z \in F$ is nontrivial in $\langle X | R_n \rangle$ but trivial in $\langle X \cup T | R_n \cup W \rangle$. Thus if $W \in \mathscr{W}$ is consistent with $G = \langle X | R \rangle$ then (by (b)) W was also consistent with $\langle X | R_n \rangle$ (the step W was considred) and so solutions for the relations W exist in G by (a). Thus G is a.c. and nontrivial and hence e.c. also.

Proof of the Theorem. To construct two c.e. groups $G = \langle X | R \rangle$ and $\overline{G} = \langle X | \overline{R} \rangle$ so that no finitely generated subgroups of them with unsolvable word problems are isomorphic, let $Y = \langle y_1, \ldots, y_n \rangle$ and $Z = \langle z_1, \ldots, z_n \rangle$ be subgroups of F. Let $Y_k = \langle y_1, \ldots, y_n \rangle \subseteq F/R_k^F$ and $Z_k = \langle z_1, \ldots, z_n \rangle \subseteq F/\overline{R}_k^F$. If $w \in Y$ let $w' \in Z$ be the result of making the substitutions $y_j \to z_j$ $(1 \leq j \leq n)$. If there exists $w \in Y$ such that $w \in R_k^F$ but $w' \notin \overline{R}_k^F$ then we can define $R_{k+1} = R_k \cup \{w\}$ and $\overline{I}_{k+1} = \overline{I}_k \cup \{w'\}$ which guarantees $(*)$: Y/R^F and $Z/\overline{R}^F$ are nonisomorphic under the map $y_j \to z_j$. We can proceed similarly if there exists $w \in Y$ such that $w \notin R_k^F$ but $w' \in \overline{R}_k^F$. So we can assume that for all $w \in Y$, $w \in R_k^F \Leftrightarrow w' \in \overline{R}_k^F$, that is, Y_k and Z_k are isomorphic under the map $y_j \to z_j$. Now put $S = \{w \in Y | R_k^F w^F \cap I_k \neq \varnothing\}$. If $S \neq Y - R_k^F$, say $w \in Y - R_k^F$, $R_k^F w^F \cap I_k = \varnothing$, then we can put $R_{k+1} = R_k \cup \{w\}$ and $\overline{I}_{k+1} = \overline{I}_k \cup \{w'\}$ which also implies $(*)$. If, however, $S = Y - R_k^F$, then we must have $Y \cap R_k^F = Y \cap R^F$ since any further relation would kill an element of $I_k \subseteq I$; thus $Y/R_k^F \cong Y/R^F$ and this group has a solvable word problem since every $R_k^F w^F$ $(w \in Y)$ can be recursively enumerated and checked to see if some member of I_k belongs to it—in which case $w \neq 1$. So by considering each sequence pair $(y_1, \ldots, y_n)$ and $(z_1, \ldots, z_n)$ $(n \geq 1)$ at some step k of the construction we guarantee that no subgroup of G with an unsolvable word problem is isomorphic to a subgroup of $\overline{G}$. It is also clear that binary branchings in the construction of the relation sets can be created to obtain $2^{\aleph_0}$ sets $R(\alpha)$ $(\alpha < 2^{\aleph_0})$ such that any two $R(\alpha)$ and $R(\beta)$ $(\alpha \neq \beta)$ have the incomparability property of R and $\overline{R}$ above.

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