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Interpolation of operators, by Colin Bennett and Robert Sharpley. Pure and Applied Mathematics, vol. 129, Academic Press, Orlando, 1988, xiv+469 pp., $69.95. ISBN 0-12-088730-4

Let us start with two L^p spaces, L^{p_0} and L^{p_1}. A function $f \in L^p$, $p_0 < p < p_1$ can be written as a sum of two functions $f = f_0 + f_1$ with $f_0 \in L^{p_0}$ and $f_1 \in L^{p_1}$. A linear operator defined on both L^{p_0} and L^{p_1} is therefore defined also on all L^p, $p_0 < p < p_1$, provided that the definition of the operator on functions in the two spaces is compatible, i.e., Tf, for $f \in L^{p_0} \cap L^{p_1}$, does not depend on whether we consider f an element of L^{p_0} or of L^{p_1}. One can then reasonably ask what properties of T on the endpoints, L^{p_0} and L^{p_1}, are transferred to the intermediate L^p. This is the simplest example which conveys the idea of interpolation theory.

Interpolation theory has been vastly generalized beyond the concrete setting described above. It is natural to replace L^p spaces by Banach spaces, but important parts of the theory have also been developed in the setting of quasi-Banach spaces (to accommodate L^p spaces with $p < 1$, and, more importantly weak-L^1 and H^p spaces with $p < 1$) and even to quasi-normed groups, and to quasi-normed semi-groups (to accommodate, say, the class of functions with monotone Fourier coefficients). In some cases the single operator T can be replaced by an analytic family of operators, with important consequences in harmonic analysis, and the two-space framework can be replaced by a family of spaces. All these generalizations are motivated by applications to various areas of analysis, principally harmonic analysis, partial differential equations, and approximation theory. Before we discuss some of these generalizations, let us return to the modest setting of L^p spaces described above.

Consider the Fourier transform of periodic functions: $f^\wedge(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,dt$. Bessel's inequality gives us

$$\left(\sum_{-\infty}^{\infty} |f^\wedge(n)|^2\right)^{1/2} \le \left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\,dt\right)^{1/2},$$

and, even more trivially, from $|e^{-int}| = 1$ we have

$$|f^{\wedge}(n)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(t)|\,dt.$$

The first inequality can be expressed by saying that the operator $^{\wedge}$ maps $L^2([0, 2\pi])$ to $L^2(\mathbf{Z})$, where the first measure space $[0, 2\pi]$ is equipped with (normalized) Lebesgue measure, and the second measure space $\mathbf{Z}$ is equipped with the counting measure: $\mu(\{n\}) = 1$. The second inequality can be expressed by saying that the operator $^{\wedge}$ maps $L^1([0, 2\pi])$ to $L^{\infty}(\mathbf{Z})$. We can now use the Riesz–Thorin interpolation theorem (Riesz, 1927; Thorin, 1939) to obtain that these two endpoints results imply that $^{\wedge}$ maps $L^p([0, 2\pi])$, for $1 \leq p \leq 2$, to $L^q(\mathbf{Z})$, where

$$\frac{1}{p} = \frac{1-\varphi}{2} + \frac{\varphi}{1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\varphi}{2} + \frac{\varphi}{\infty},$$

which gives $1/p + 1/q = 1$. The theorem also gives information about the norm of the operator on the intermediate spaces, which, since the norm of $^{\wedge}$ is one at both endpoints yields:

$$\left(\sum_{-\infty}^{\infty} |f^{\wedge}(n)|^q\right)^{1/q} \leq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p\,dt\right)^{1/p}.$$

This is the Hausdorff–Young theorem (1912–1923). W. H. Young proved the theorem for even integers q and Hausdorff extended it to all $q > 2$. The proof, naturally, was not the one above.

The interpolation proof reveals the limitations of the theorem. One cannot hope to deduce from the theorem any conclusion which depends on any deeper property of the trigonometric system than the fact that it is a uniformly bounded orthonormal system. Even the completeness of the system does not enter the picture.

The example above illustrates the advantages of interpolation results. In the first instance interpolation theory offers easy proofs of some theorems. More importantly, the easier proofs enable us to gauge properly the significance of these theorems.

We should point out that in this review "Riesz" refers to Marcel Riesz. The Riesz–Thorin interpolation theorem was proved by Riesz (1927) using truly elementary tools: determining necessary and sufficient conditions for equality in Hölder's inequality for sequences. Thorin's proof (1939) came from left field. One constructs an analytic function on $0 \leq \mathrm{Re}(z) \leq 1$, $f(\cdot, z)$, whose values on $\mathrm{Re}(z) = 0$, $f(\cdot, it)$, are L^{p_0} functions, and on $\mathrm{Re}(z) = 1$,

$f(\cdot, 1+it)$, are L^{p_1} functions. $Tf(\cdot, z)$ are again analytic functions, whose behavior on $\mathrm{Re}(z) = 0$ and on $\mathrm{Re}(z) = 1$ is controlled by the boundedness of T on the endpoint spaces, L^{p_0} and L^{p_1}. One then applies Hadamard's three line theorem to control the size of the function at $z = s$, $0 < s < 1$, which gives the right estimate on L^p, for the intermediate p,

$$\frac{1}{p} = \frac{1-s}{p_0} + \frac{s}{p_1}.$$

We have used the Fourier transform to motivate the Riesz–Thorin interpolation theorem. Let us now consider the Hilbert transform, defined by

$$Hf(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|t|>\varepsilon} \frac{f(x-t)}{t}\, dt.$$

Starting with f, if we let $F = P_y * f$ be the harmonic extension of f to the upper half-plane, and then let G be the harmonic conjugate of F, i.e. $F + iG$ is analytic in the upper half-plane, then it turns out that $\lim_{y\to 0^+} G(x, y) = Hf(x)$. It is impossible to overstate the importance of this operator, connecting as it does real analysis, harmonic analysis, complex analysis, and, through its generalizations to $\mathbf{R}^n$, partial differential equations. It has also served as a motivation for some of the most fundamental work in interpolation theory.

It is not hard to see that $H: L^2 \to L^2$. The proof that $H: L^p \to L^p$ for other values of p, using interpolation, is rendered difficult by the fact that the usual endpoints, L^1 and L^∞, are not available for the Riesz–Thorin theorem. It is not hard to see that H does not map either of these spaces into itself. From the complex analysis interpretation of the Hilbert transform we get

$$H\left(\frac{1}{1+x^2}\right) = \frac{x}{1+x^2}. \quad \frac{1}{1+x^2} \in L^1, \quad \text{but } \frac{x}{1+x^2} \notin L^1.$$

For large x, $x/(1+x^2)$ is like $1/x$.

Instead of L^1 boundedness H has the weaker property $|\{|Hf| > \lambda\}| \le C\cdot\|f\|_1/\lambda$, i.e. H maps L^1 to weak-L^1. Riesz's beautiful proof (1927) of $H: L^p \to L^p$ uses the complex analysis interpretation of H to prove the L^p norm inequality for all even integer values of p. Next the Riesz–Thorin interpolation theorem is used to get the estimate or all $p > 2$, and, finally, since the dual operator to H is $-H$, we get the theorem for $1 < p \le 2$ as well. (This is one of two proofs Riesz gave of the continuity of H. In the other proof, also published in 1927, the extension from even integers to other values of p is achieved by complex analysis considerations.)

Alongside the Riesz–Thorin theorem, the other classical interpolation theorem is due to Marcinkiewicz (1939). It was motivated by the idea of proving the L^p continuity of H by interpolating between its L^2 continuity and the inequality $|\{|Hf| > \lambda\}| \leq C \cdot \|f\|_1/\lambda$. Marcinkiewicz' idea was to decompose $f \in L^p$ into two functions, $f = f^y + f_y$, where

$$f^y = \begin{cases} f \text{ if } |f| \geq y \\ 0 \text{ otherwise}. \end{cases}$$

$f_y \in L^2$, and $f^y \in L^1$. $Hf = Hf^y + Hf_y$, and the behavior of H on L^2 and on L^1 is used to control Hf_y and Hf^y. The level of the cut, y, is then varied, and a precise calculation of the resulting norm inequalities gives the L^p result.

The examples outlined so far point to the great usefulness of interpolation theorems. However, proving ad hoc interpolation theorems for each needed case is itself not an efficient method. It is as if telephone service were provided by connecting each customer by a special line to each other customer. It is far more efficient to connect each customer to a central exchange. The construction of such exchanges in the late 1950's marked the beginning of the modern theory of interpolation of operators. In essence, instead of interpolating the operators, we interpolate the spaces. Given spaces which are compatible—this has of course a precise technical meaning, but the idea is to axiomatize the fact that the algebra and topology in L^p spaces for different values of p are consistent—we construct new spaces which have the interpolation property. This means that if we are given two sets of endpoint spaces, $\overline{A}$ and $\overline{B}$, and if we construct the interpolation spaces for both sets, $\overline{A}$ (parameters) $\overline{B}$ (parameters), then any linear operator (in real interpolation theory the operators need not be linear; weaker conditions suffice) which is continuous from $\overline{A}$ to $\overline{B}$ will be continuous from $\overline{A}$ (parameters) to $\overline{B}$ (parameters). For the results to be interesting, the constructed spaces should, of course, have intrinsic characterizations, and the identification of the interpolation spaces should incorporate the known interpolation theorems. For a simple example, the interpolation spaces between L^p spaces should, for some values of the parameters, yield either the

Riesz–Thorin or the Marcinkiewicz interpolation theorem.

We will confine our attention to the two major methods of constructing interpolation spaces, the real method, which is the abstract version of the Marcinkiewicz theorem, and the complex method, which is the abstract version of the Riesz–Thorin theorem.

In the real method, a family of norms $K(t, a; A_0, A_1)$ (the word is used loosely; they could be a great deal less than fully fledged norms) depending on a parameter $t > 0$, is defined on $a \in A_0 + A_1$ (the ability to add elements of the two endpoint spaces follows from the compatibility requirements on $\overline{A}$). The interpolation spaces are then defined by requiring that, as a function of t, K satisfies some integrability conditions. This corresponds to the varying level of cuts in the Marcinkiewicz theorem. The information carried by the behavior of $K(t, a; A_0, A_1)$ can also be expressed in terms of the important E-functional or approximation functional $E(t, a; A_0, A_1)$ which measures how closely (in the A_1-norm) we can estimate a by an element whose A_0-norm is less than t. The intuitive simplicity of E makes its calculation very easy in some cases.

In the complex method, the interpolation spaces consist of the values in the interior of a domain, of Banach space valued analytic functions which on the boundary of the domain belong to the endpoint spaces, the interpolated family. (In an important special case the interpolated family consists of only two spaces A_0, A_1.) This corresponds to the Thorin construction described above.

Each method has its own advantages. The real method is in many cases richer. Consider the simple example of the Fourier transform. As we saw above, the complex method, when applied to the simple endpoint results yields the Hausdorff–Young theorem,

$$\left(\sum_{-\infty}^{\infty} |f^\wedge(n)|^q\right)^{1/q} \leq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p\, dt\right)^{1/p}.$$

The real method when applied to the same endpoint estimates, without any additional input from harmonic analysis work, gives the stronger Paley's theorem:

$$\left(\sum_{n=1}^{\infty} [f^\wedge(n)]^{*p} n^{p-2}\right)^{1/p} \leq C(p)\left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p\, dt\right)^{1/p}$$

where $[f^\wedge(n)]^*$ denotes the nonincreasing rearrangement of $|f^\wedge(\cdot)|$.

In the case of the Hilbert transform, the complex method can handle the weak-type inequality at $p = 1$: $|\{|Hf| > \lambda\}| \leq C \cdot \|f\|_1/\lambda$, although not by the methods discussed in this book, but the conclusion is not satisfactory, since we fail to get the L^p continuity of the operator. Another important advantage of the real method: it handles quasi-norms (triangle inequality weakened to $\|a + b\| \leq C(\|a\| + \|b\|)$, in some instances homogeneity too is sacrificed) as easily as it does norms. To people brought up in functional analysis, this might seem a curiosity, of no great importance. Note however that the range space of the Hilbert transform acting on L^1, the space weak-L^1 as it is called in the classical literature, or $L(1, \infty)$ in the modern one, the space of functions satisfying $\mu\{|f| > \lambda\} \leq A/\lambda$, is quasi-normed (in fact, for every $C > 1$ a quasi-norm on it can be found which satisfies a C-triangle inequality) but is not normable. Moreover, the generalization enables us to incorporate in one interpolation scale H^p spaces for $0 < p < \infty$.

Finally, real interpolation theory is easily applicable to some nonlinear operators. The original paper of Marcinkiewicz already announces his theorem for quasi-linear operators. This, again, is important in many applications.

The complex method, as one may expect where analytic functions come into play, is much more rigid. This has the advantage of providing sharper norm estimates in some applications. Furthermore, the rigidity of the method is a tool by which properties of an operator on an interpolation space spread to nearby spaces. The complex method also enables us to vary the operators analytically even as we vary the spaces. To emphasize the point, consider: if T_{it} are continuous on L^{p_0} and T_{1+it} are continuous on L^{p_1}, then, if T_z form an analytic family of operators, we have the continuity of T_s on $L^{p(s)}$. Moreover, the method permits us to have not only two endpoint spaces, but whole families of them. On the other hand, the extension of the theory to quasi-normed spaces is very problematic. At the core of the difficulties is the failure of the maximum principle for functions taking their values in quasi-normed spaces. The best we have available at this time are results obtainable with the interplay of the complex and the real methods.

Both methods have rich theories. For example, we have reiteration theorems, which say that if we interpolate between spaces which are themselves interpolation spaces, the interpolation spaces we get will include all the right interpolation spaces between the original endpoint spaces. We also have information about the interpolation of spaces which are duals of a given set of endpoint spaces. The two methods collaborate in that we know what are the interpolation spaces in the complex method of endpoints which are themselves interpolation spaces in the real method, and vice versa.

In the context of real interpolation theory one should make special mention of recent work on "K-divisibility." We have mentioned before that the real method is an abstract version of the Marcinkiewicz interpolation theorem, with the K-functional providing the analogue of the variable-level cut of functions. K-divisibility pushes this analogy further and provides powerful tools for extending results hitherto available only in the setting of $L(p, q)$ spaces or weighted L^p spaces to real interpolation scales of other quasi-Banach spaces. Roughly speaking, the K-divisibility theorem enables us to manipulate the functions $K(t, a; A_0, A_1)$ with almost as much freedom as if they were arbitrary functions of t on $(0, \infty)$, and almost ignore the fact that they are determined by elements $a \in A_0 + A_1$.

If we continue with the analogy of interpolation theory and telephone communications, the work described above is the construction of the exchanges. But another no less important part of the theory is the linesman's job of connecting customers to the exchanges: finding the interpolation spaces for given endpoint spaces. The benefits of such results are clear: operators interpolate (and sometimes extrapolate), giving new and interesting results. But there can be another payoff which may even transcend these applications. The calculation of the interpolation spaces involves a detailed analysis of the spaces in question, leading to a better understanding of their properties.

The book *Interpolation of Operators* is a graduate text, where the real method of interpolation is motivated, as the subject itself was, by central problems in harmonic analysis. One cannot say enough in praise of such a program. It gives the student a feeling for the development of the field, a sense of what is central and what is peripheral. It encourages the student to ask the right questions when embarking on his own research. The value of the book as an invitation to the field is enhanced by a felicitous writing style, and the thoughtful selection of some elegant applications from the literature.

The authors have also made several choices which might be questioned. We shall mention just two. One is the inclusion, at the beginning of the book, of a detailed exposition of Banach function spaces and rearrangement invariant spaces. The abundance of material on these subjects has crowded out some central topics in interpolation theory, such as the interpolation of dual spaces. Moreover, the student who wishes to learn real interpolation will have to be patient. The K-functional is not introduced until page 293. (The approximation functional is not mentioned at all.) A second major decision taken by the authors was to present real interpolation theory only for Banach spaces. This, particularly in the context of real interpolation and in a book interested in applications, is somewhat surprising. As we have endeavored to explain, the more general setting does not complicate the theory, and is most useful in applications.

One small carping. The operators admitted in the general theory (Chapter 5), are required to be linear. This leaves the reader with the impression that real interpolation theory does not encompass the Marcinkiewicz interpolation theorem, where the operators are quasi-linear. A definition of admissible operators in real interpolation which, when applied to the $L(p, q)$ scale, yields the Marcinkiewicz theorem in full, has been known at least since 1969.

The authors should be congratulated on this attractive book. It could serve as an excellent text on Banach function spaces. Supplemented by the older book by Bergh and Löfström, and by various recent results in the literature, it can be the basis for an excellent course on interpolation theory as well. For experts in interpolation theory it offers an exposition of several elegant applications of the theory.

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