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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 23, Number 1, July 1990

Algebraic homotopy, by H. J. Baues. Cambridge University Press, Cambridge, New York, 1989, 466 pp., $89.50. ISBN 0-521-33376-8

The notion of classification of structure arises in many areas of mathematics, and a common classification is "up to homotopy," or in terms of "deformation." For this reason, techniques of homotopy theory, and in particular the fundamental group and higher homotopy groups, are important and have been applied across a range of mathematical disciplines.

Algebraic Homotopy, which we refer to as AH, has in the Introduction the following quotation from J. H. C. Whitehead's address to the International Congress of Mathematicians at Harvard in 1950 [W 7]:

> In homotopy theory, spaces are classified in terms of homotopy classes of maps, rather than individual maps of one space in another. Thus, using the word category in the sense of S. Eilenberg and Saunders Mac Lane, a homotopy category of spaces is one in which the objects are topological spaces and the 'mappings' are not individual maps but homotopy classes of ordinary maps. The equivalences are the classes with two-sided inverses, and two spaces are of the same homotopy type if and only if they are related by such an equivalence. The ultimate object of *algebraic homotopy* is to construct a purely algebraic theory, which is equivalent to homotopy theory in the same sort of way that 'analytic' is equivalent to 'pure' projective geometry.

One reason for attempting such a problem is of course simply to understand the homotopy theory of polyhedra. Much of Baues's work, in AH and elsewhere, is guided by the overall aim of developing algebraic machinery which would, in principle and in low-dimensional cases, allow for the translation into algebraic problems all of the basic geometric questions in the area, and allow for many calculations. Many of these problems occur in geometric situations which are of interest in other areas of mathematics. For example, a new calculation given on p. 291 of AH, of the group of homotopy classes of homotopy equivalences of a connected sum $(S^1 \times S^3)\#(S^2 \times S^2)$, is relevant to smoothing theory.

Mathematics of this kind is likely to be interesting not only for its own sake or for its immediate applications, but also for the insights and techniques which can become appropriate to other areas of mathematics. For example, the relation found by Hopf in 1941 between the second homology group of an aspherical space and its fundamental group, the so-called Hopf formula $H_2G = (R \cap [F, F])/[R, F]$, was one of the starting points of homological algebra, without which the solutions of a number of important problems, for example the Weil conjectures, would not have been conceivable. Whitehead's study of the algebra underlying geometric collapsings led to his formulation of the K_1-group of a ring [W 5]. Thus the study of the formalities underlying homotopy theory has revealed a range of new techniques which were found suggestive of development and capable of wide applicability.

The features of AH which make it an original and important contribution to the literature on homotopy theory are as follows: (i) the generality of the methods; (ii) the global approach to the study of homotopy categories and of approximations to homotopy theory; and (iii) the extension of a range of ideas developed by Whitehead about 1950, and not previously available in texts.

(i) Generality of the methods. Experience has shown that to study a homotopy category, it is important to study the original category and the homotopies, and also higher homotopies. The notion of a category with a notion of homotopy arises in so many examples and guises that a number of axiomatizations have been introduced with a view to yielding one theory covering all cases (Quillen, K. S. Brown, Kamps,...). Baues uses a modification of Quillen's methods, which he calls a *cofibration category*. This is a category with a notion of *weak equivalence* and of *cofibration*, satisfying suitable axioms. The Introduction to AH states:

> Therefore, the unification due to the abstract development possesses major advantages: *one proof replaces many*; in addition, an interplay takes place among the various applications. This is fruitful for many topological and algebraic contexts.

As a consequence, roughly two thirds of AH deals either with the general situation, or with examples other than topological spaces. Even the third or so on the latter case considers often the case of spaces over or under another space, so that the general categorical discussion can be applied. One of the surprises is that the final chapter, which deals with Whitehead's classification of simply connected four-dimensional polyhedra in terms of his "Γ-sequences" [W 6], does this in a context that applies not only to CW-complexes but also to localized CW-complexes, to chain algebras, and to chain Lie algebras. G. W. Whithead's famous result on the nilpotency of the group of homotopy classes $[\Sigma X, Y]$ for finite-dimensional X is here proved in a general cofibration category, and so is again available in the above examples.

(ii) The global approach to approximation to homotopy categories. The study of the homotopy category $\text{Ho}(C)$ of a cofibration category C can be very difficult, but it may be possible to study successfully some of its subcategories. For example, in the topological case there is the full subcategory $\mathbf{M}^n$ of Ho(Top) whose objects are the Moore spaces $M(A, n)$ for various Abelian groups A, i.e., simply connected spaces with one nonvanishing homology group A in dimension n. For $n \geq 3$, Baues describes this category as a *linear extension of categories* over the category of Abelian groups. This is related to work of M. G. Barratt in 1954 describing this category in terms of generators and relations.

This notion of linear extension of categories is a key new idea in AH. It generalizes the notion of an extension $A \rightarrowtail E \twoheadrightarrow G$ of groups where A is a G-module. (It is well known that there is a conflict in the literature as to whether such an extension is *of* A or *of* G; Baues calls this an extension of G.) Linear extensions of categories should be expected to have general algebraic applications. They are used in Chapter V of AH to examine a set of three approximations to a category of maps of mapping cones, and in Chapter VI to examine the pieces of a *tower of categories* approximating the homotopy theory of CW-complexes. This general idea of a tower of categories looks to be proving important. For example, it is used in Chapter VI to extend work of Whitehead in [W 4] on the relation between crossed chain complexes (see below) and chain complexes with operators.

This idea of finding approximations to the homotopy category of CW-complexes goes back to Whitehead. He and Spanier introduced *stable homotopy theory*, which has been actively developed by many homotopy theorists.

Other approximations considered by Whitehead were obtained by making restrictions on the dimensions and connectivities of the spaces. A space X is called *n-coconnected* if it is connected and $\pi_i X = 0$ for $i \geq n$. The homotopy category of $(n+1)$-coconnected CW-complexes is called the *n-type.* Part of the problem of algebraic homotopy is to give algebraic equivalents of n-types, or of $(n-1)$-connected $(n+k)$-types, or of A_n^k, the homotopy category of $(n-1)$-connected polyhedra of dimension not greater than $(n+k)$.

Chapter V of AH gives new descriptions of the categories A_n^2 $(n \geq 2)$ as linear extensions of categories. This allows for computations of homotopy classes of maps. Chapter V also describes linear extensions of a category C in terms of the second cohomology of C. So there are intriguing questions, discussed elsewhere by Baues, of the determination of the cohomology classes of the extensions which arise in the description of homotopy categories. That really is global homotopy theory!

(iii) The expression of Whitehead's work. AH is the first text to give extensive expression to one major part of the wide range of ideas which were developed by Whitehead in this area and published in the years around 1950.

Two of Whitehead's papers of that time are particularly well known. "Combinatorial homotopy I" [W 3] laid the foundations of CW-complexes. These complexes give a convenient method for handling combinatorial decompositions of a space as the union of cells, in such a way as to allow proofs by induction over dimension. "Simple homotopy types" [W 5] laid the basic methods of algebraic K-theory and their geometric applications. These two papers, with a third, "Combinatorial homotopy II" [W 4], rewrote and extended work Whitehead published about 1940, which itself showed a mastery of the combinatorial methods of the 1930s and an amazing insight into the way these methods could be extended.

"Combinatorial homotopy II" (hereinafter referred to as CHII) is clearly the odd man out in terms of the attention it has received. However, it does contain an important set of algebraic tools.

One of these, the cellular chains of $\pi_1(X)$-modules $C_*(\tilde{X})$ of the universal covering space $\tilde{X}$ of a connected CW-complex X, goes back to Reidemeister. It has been widely used both in simple homotopy theory, where it is an essential tool, and in low-dimensional topology. The paper CHII contains results which imply a useful and not so well-known homotopy classification theorem, namely a bijection of homotopy classes

$$[X, Y] \to [C_*(\tilde{X}), C_*(\tilde{Y})]$$

when X is n-dimensional and $\pi_i Y = 0$ for $1 < i < n$. This contains in essence later work of Olum. See [E] for a recent exposition of applications. The result is a corollary of (4.9) and (6.15) of Chapter VI of AH.

This Chapter VI extends basic results of CHII on what Whitehead called *homotopy systems.* These are now, after dropping the freeness assumptions used by Whitehead, called *crossed complexes* or, as in AH, *crossed chain complexes.* See [Lu], [B 1] for discussions of their occurrence. In the approach of AH, the homotopy category of free crossed chain complexes is the first level in the tower of approximations to the homotopy category of CW-complexes, in which successive levels are related in terms of linear extensions of categories.

A crossed chain complex is like a chain complex C of G-modules for a group G, except that the part $C_2 \to C_1$ has cokernel G and is a non-Abelian structure known as a crossed module. This element of structure has a wide importance, but is not so well known even 44 years after its introduction by Whitehead in [W 2]. We therefore give a definition and a wider perspective than in AH.

A *crossed module* (*of groups*) consists of a morphism of groups $\mu: M \to P$ together with an action of P on (the right of) M, written $(m, p) \mapsto m^p$, satisfying the two rules:

$$\text{CM1)}\ \mu(m^p) = p^{-1}(\mu m)p,$$

$$\text{CM2)}\ m^{-1}m_1 m = m_1^{\mu m},$$

for all $m, m_1 \in M, p \in P$.

Examples of crossed modules are: a normal subgroup M of P with the conjugation action; a P-module M, so that μ is constant;

the inner automorphism map $M \to \operatorname{Aut} M$ for any group M; an epimorphism $M \to P$ of groups with central kernel. Another important example is the free crossed module $\mu: C(R) \to F(X)$ defined by a presentation (X, R) of a group G. Here μ has cokernel G and kernel the G-module of *identities among the relations* R. So this example should be thought of as yielding for G the beginning of a *free resolution* which includes non-Abelian information and which arises naturally in terms of chains of syzygies for the presentation (see the surveys [B-Hu] and [B 1]).

As might be expected of any good notion, crossed modules occur in a variety of guises and of analogues. The guises include: *group objects in the category of groupoids* [B-S 1]; *double groupoids with connection* [B-S 2]; *simplicial groups whose Moore complex is of length* 1; and cat^1*-groups* [L]. The analogues include: crossed modules of Lie algebras; of associative algebras; and of Jordan algebras; indeed, crossed modules may be expected to arise in any algebraic situation in which equivalence relations are determined by kernels. (See for example [P].) All this gives additional point to the occurrence of crossed modules (of groups) in the homotopical situation.

The geometric example is the *fundamental crossed module* of a pair of pointed spaces (X, A, x), or of a pair of pointed objects in a cofibration category, namely the boundary map $\partial: \pi_2(X, A, x) \to \pi_1(A, x)$ of the second relative homotopy group, with its usual action of $\pi_1(A, x)$ [W 2]. Because of this example, one should think of a crossed module $M \to P$ as a kind of *two-dimensional group* with a one-dimensional part, the group P, and a two-dimensional part, the group M with the action of P. This example arose out of Whitehead's investigations into *adding relations to homotopy groups* [W 1,2,4], which attempted to determine how the homotopy groups of a space are influenced by the addition of a cell.

There is an extraordinary point to be made here. This last investigation of Whitehead can be regarded as an attempt to generalize to higher dimensions the Van Kampen theorem for the fundamental group. It tends to be forgotten that a definition of higher homotopy groups was given by Čech in 1932, in a paper submitted to the International Congress of Mathematicians at Zurich. However, it was quickly proved that these groups were Abelian, and Alexandroff and Hopf persuaded Čech to withdraw his paper, so that only a small paragraph appeared in the proceedings [C]. It seems that they believed that the proposed groups had to be the same as the homology groups. Otherwise, how could the higher-dimensional invariant be apparently simpler than the one-dimensional invariant? In later years, both Alexandroff and Hopf frankly admitted the mistake that had been made [A].

Now the argument that the higher homotopy groups are Abelian is a special case of the argument that the second relative group $\pi_2(X, A, x)$ satisfies axiom CM2 for a crossed module. Further, crossed modules model all homotopy 2-types, as shown by Mac Lane and Whitehead in [M-W]. (AH is the first text to give a proof of this result.) The clincher argument for the role of crossed modules as two-dimensional groups is that they satisfy a version of the Van Kampen theorem, namely that the fundamental crossed module functor on pairs of pointed spaces preserves certain colimits [B-H 1]. This enables new calculations to be made not only of both relative and absolute second homotopy groups, but even of homotopy 2-types [B-H 1, B 2]. It also has as a corollary Whitehead's result (Theorem VI.1.12 of AH, which is crucial for much of the work of Chapter VI of AH) that the second relative homotopy group $\pi_2(X \cup \{e_\lambda^2\}, X)$ is a free crossed $\pi_1(X)$-module [W 1,2,4]. I remember Whitehead remarking to J. Milnor that the early workers in homotopy theory were fascinated by the action of the fundamental group. All this suggests that the initial embarrassment with the Abelian nature of homotopy groups, though it has come to seem a quirk of history, in fact represented an honest and useful mathematical reaction. The algebraic analysis of non-simply connected *homotopy types* requires non-Abelian structures, of which groups and crossed modules represent the first two stages. Indeed, it can be argued that the natural progression is not from the fundamental group, but from the fundamental groupoid to *higher homotopy groupoids*, of which the *n-cat-groups* of [L] form a principal example, and which are "highly non-Abelian" structures (see [E-S]).

The generalized Van Kampen theorems ([B-H **1,2**], [B-L]) are proved by methods different from those in AH, and do not appear there. It is intriguing that crossed modules are basic to both approaches, and that Whitehead's result on free crossed modules is stated but not proved in AH. This suggests that relating these different methods could prove productive.

I have spent a lot of space on one aspect of AH, because of its importance and general unfamiliarity. Readers of AH will also welcome the accounts of, for example, rational homotopy theory, the homotopy spectral sequence, local cohomology, small models, and obstructions to finiteness. The book contains much new material, is a pointer to future research, and will also form an excellent framework for various courses on homotopy theory, allowing students to consider the literature from a new and worthwhile perspective.

Acknowledgments

I would like to thank G. Chogoshvili, E. Dyer, and Z. Groschaftová (Professor Čech's daughter) for information on the reactions to Čech's work on homotopy groups.

It is a pleasure to be making this tribute to Whitehead's papers CHI and CHII in this journal.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 23, Number 1, July 1990

0273-0979/90 $1.00 + $.25 per page

Andreotti-Grauert theory by integral formulas, by Gennadi M. Henkin and Jürgen Leiterer. Progress in Mathematics, vol. 74, Birkhäuser, Boston, Basel, Berlin, 1988, 270 pp., $44.90. ISBN 0-8176-3413-4

Experts in multidimensional complex analysis will find the title of this monograph sufficiently informative, but most other mathematicians will probably feel lost, and perhaps not bother to look closer at this book. That would be regrettable, because what is before us is the first attempt to make accessible to a wider audience the deep work of A. Andreotti and H. Grauert published in