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A NONLINEAR EXTENSION OF THE BOREL DENSITY THEOREM: APPLICATIONS TO INVARIANCE OF GEOMETRIC STRUCTURES AND TO SMOOTH ORBIT EQUIVALENCE

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Let G be a connected semisimple Lie group with no compact factors and finite center and let Γ be a lattice in G (i.e. a discrete subgroup such that G/Γ has a finite invariant measure). Let π be a representation of G on some vector space V. Borel [Bo] proved that if π is a rational representation and V is finite dimensional then every Γ-invariant line in V is G-invariant; in fact, this is equivalent to saying that Γ is Zariski dense in G. On the other hand, if we allow V to be infinite dimensional but we require π to be unitary, Moore [M] proved that every Γ-invariant vector $v \in V$ is G-invariant; this is true for Γ not necessarily discrete, as long as it is not compact. However, the same result is far from being true for any infinite dimensional representation. Here we announce the proof of an extension of Borel's theorem in two different directions: one involving nonlinear actions, the other involving some particular infinite dimensional linear representations which arise naturally from purely geometric considerations.

Let G, Γ be as above with the further assumption that Γ is irreducible (i.e. we want to eliminate the case in which $\Gamma = \Gamma_1 \times \Gamma_2 \subset G_1 \times G_2 = G$) and let H be a real algebraic group. Let M be a smooth manifold and let $P \to M$ be a principal H-bundle on which G acts by automorphisms. Suppose that X consists of the real points of a variety defined over $\mathbf{R}$ on which H acts algebraically and let $E \to M$ be the bundle with fiber X associated to P. Then we have the following result:

Theorem 1. *If every orbit in M with compact stabilizer is not locally closed, then every measurable Γ-invariant section of E is G-invariant.*

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Note that if X is a finite dimensional vector space and the H-action is linear, in which case E is a vector bundle, then the space of sections is an infinite dimensional vector space on which G acts linearly. If, moreover, M is a point then our result reduces to the Borel density theorem. Furthermore, if π is an admissible finitely generated representation of G on a Banach space V, the result above implies that every C^∞-vector in V which is Γ-invariant is also G-invariant. This is because the space of C^∞-vectors for such a representation can be continuously and G-equivariantly embedded in the space of C^∞-vectors of a representation induced from a finite dimensional representation σ on X of the minimal parabolic subgroup P, that is in the space of C^∞-sections of the bundle over $M = G/P$ associated to $G \to G/P$ with fiber X [Wa].

A result closely related to Theorem 1 can be stated in terms of geometric structures. Namely, with the same hypotheses on G, Γ, M and on the orbits in M as above, if $n = \dim(M)$ and $H \subseteq \mathrm{GL}(n, \mathbf{R})$, we have the following:

Theorem 2. *If H is algebraic, then every Γ-invariant H-structure on M is G-invariant.*

Remark. If M is an n-dimensional compact manifold, H, G, and Γ are as above with the further assumptions that $H \subseteq \mathrm{SL}(n, \mathbf{R})$ and that each factor in G has $\mathbf{R}$-rank at least 2, then it has been conjectured [Z 2] that if Γ preserves an H-structure on M then either there is a Lie algebra embedding $\mathfrak{g} \hookrightarrow \mathfrak{h}$ or there is a smooth Γ-invariant Riemannian metric on M. Partial results have been obtained in this direction for some particular classes of actions [Z 2]; however, since the existence of such an embedding is known when the H-structure is preserved by G [Z 3], our result gives further evidence to the above conjecture, as long as the Γ-action can be extended to a G-action, without any other specific requirement on the action.

Before sketching the proofs of our results we want to give some examples to show that the hypotheses of these theorems are sharp. Observe first that if H is not algebraic then Theorem 2 need not hold: for example let G, G' be simple noncompact algebraic groups and let G be embedded in G' so as to act ergodically on $G'/\Gamma' = M$, where Γ' is a lattice in G'. Assume for simplicity that G has trivial center. The tangent bundle to the orbits in M is smoothly equivalent to $M \times \mathfrak{g}$, where $\mathfrak{g}$ is

the Lie algebra of G, and the action of G by automorphisms is given by $g \cdot (m, v) = (g \cdot m, \mathrm{Ad}(g)v)$. Hence, since $G \simeq \mathrm{Ad}(G)$, there is a natural reduction P to G of the frame bundle $M \times \mathrm{GL}(\mathfrak{g})$ associated to $M \times \mathfrak{g}$; since for every $g, g_0 \in G$ we have $g_0 \cdot (m, g) = (g_0 \cdot m, g_0 g)$, then any closed subgroup Γ of G gives a Γ-invariant reduction of P to Γ. In particular suppose Γ is a lattice, hence not algebraic: then if such a reduction to Γ were also G-invariant this would imply the existence of a G-map $\varphi: G'/\Gamma' \to G/\Gamma$ which is measure preserving (by the uniqueness of the G-invariant measure on G/Γ) and hence [Wi 1], of a continuous surjective homomorphism $\sigma: G' \to G$, which is impossible if G' has dimension higher than G, since G' is simple.

Notice moreover that the hypothesis on the orbits in M is also essential, at least in the case in which M is not compact. In fact if G acts on itself by translations, the lifting to G of any vector field on G/Γ, is Γ-invariant but not G-invariant. In the case in which M is compact, the question is a little more delicate. In fact, if Γ is any lattice in $G = \mathrm{SL}(2, \mathbf{R})$ acting by fractional linear transformations on S^2, it can be proven that there are nontrivial continuous vector fields on S^2 which are Γ-invariant but not G-invariant (here, of course, the problem is that the hemispherical orbits in S^2 have $SO(2, \mathbf{R})$ as stabilizer); however, one can also show that there are no such vector fields which are also smooth, and therefore one might conjecture that if M is compact and we restrict our attention to smooth structures, the hypothesis on the orbits can be removed.

Comment on the Proof of Theorem 1. After a measurable trivialization of the H-structure $P \to M$, the G-action on P yields a measurable cocycle $\alpha: M \times G \to H$. With the corresponding trivialization $E \simeq M \times X$, if $F(M, X)$ denotes the space of measurable functions from M to X with the α-twisted action of G [Z 1], a Γ-invariant section of E corresponds to an α_Γ-invariant function $f \in F(M, X)$, where $\alpha_\Gamma = \alpha|_{M \times \Gamma}$. On each ergodic component S of M the tameness of the H-action on X and the ergodicity of the restriction of the action to Γ imply that f takes values in some H-orbit, say $H \cdot x_s$ for some $x_s \in X$. Then if H_{x_s} is the stabilizer of $x_s \in X$, and hence H_{x_s} is a real algebraic subgroup of H, we can consider $f|_S$ as a function in $F(S, H/H_{x_s}) \hookrightarrow F(S, \mathbf{P}^{n-1}(\mathbf{R}))$ (the last inclusion being given by

the identification of H/H_{x_s} with an orbit in some projective space $\mathbf{P}^{n-1}(\mathbf{R})$ via a rational representation [Ch]). We can then choose a possibly different trivialization in such a way that saying that $f|_S$ is α_Γ-invariant amounts to saying that, $\alpha_\Gamma(S \times \Gamma) \subseteq H_{x_s}$. The proof of the theorem will then be completed by using the following general result on cocycles to study the action of some particular cocycles on subspaces of the projective space.

Definition. [Z 1] (i) If α is any cocycle of an ergodic action into an algebraic group H, the algebraic hull of α is the smallest subgroup of H, unique up to conjugacy, in which a cocycle equivalent to α takes values. (ii) If $L \subset \mathrm{Diff}(M)$, the algebraic hull of the L-action on M is the algebraic hull of the action of L by automorphisms of the frame bundle on M (notice that this is well defined since different measurable trivializations give equivalent cocycles).

Theorem 3. *Let G and Γ be as above, S be an ergodic G-space not essentially isomorphic (as a Borel G-space) to G/G_0 with G_0 compact and let $\alpha: M \times G \to H$ be a cocycle. Then the algebraic hulls of α and $\alpha|_{M\times\Gamma}$ are the same.*

Comment of the Proof of Theorem 2. Since H-structures are in bijective correspondence with sections of the bundle $P(M)/H \to M$, where $P(M)$ is the frame bundle associated to the tangent bundle, the result follows from Theorem 1 since in this case we can choose the stabilizers H_{x_s} to be the same for each ergodic component.

The theorem above on algebraic hulls can be used also in a completely different direction to derive results on smoothly orbit equivalent actions of groups. Let M_i, $i = 1, 2$ be smooth manifolds on which H_i acts by diffeomorphisms; we say that the actions of H_1 and H_2 are smoothly orbit equivalent if there exists a diffeomorphism $\theta: M_1 \to M_2$ which preserves the orbits, namely $\theta(h_1 m_1) = h_2 \theta(m_1)$. If the actions are essentially free, then $\lambda: M_1 \times H_1 \to H_2$, $\lambda(m_1, h_1) = h_2$, defines a cocycle. If $H_1 = H_2$ and we can take $h_1 = h_2$, then we say that the actions are smoothly conjugate.

Theorem 4. *Let G be a semisimple noncompact Lie group with no compact factors and finite center, $\Gamma \subset G$ an irreducible lattice and $H \subset G$ a noncompact closed subgroup. Then the algebraic hull of the H-action on G/Γ is the algebraic hull of* $\mathrm{Ad}_G(H)$.

Corollary 5. *Let G, Γ be as above and let H be a noncompact Lie group acting on G/Γ via two different embeddings $\pi_1, \pi_2 : H \to G$ with closed images. If the actions of H on G/Γ given by π_i's are smoothly conjugate then the algebraic hulls of $\mathrm{Ad}_G(H_1)$ and $\mathrm{Ad}_G(H_2)$ are conjugate as subgroups of $\mathrm{GL}(\mathfrak{g})$.*

Orbit equivalence phenomena for group actions have been studied by several authors ([Be], [PZ], [Wi 2], [Z 1], [Z 4]) using different approaches; however, the result we want to state here gives an answer in some cases, for instance on the group $H = ax + b$ of affine motions, which does not seem to be accessible so far by other techniques.

Theorem 6. *Let H_1, H_2 be connected k-dimensional Lie groups acting by diffeomorphisms on smooth n-dimensional manifolds M_1, M_2. Suppose that the H_i-actions are essentially free and smoothly orbit equivalent. Then the algebraic hulls of the derivative cocycles in the direction normal to the orbits are conjugate as subgroups of $\mathrm{GL}(n-k)$.*

Example. Let $H = \mathbf{R}^\times \ltimes \mathbf{R}$ be the group of the affine motions of the line acting on $\mathrm{SL}(3, \mathbf{R})/\Gamma$, where $\Gamma \subset \mathrm{SL}(3, \mathbf{R})$ is any lattice, via the two embeddings

$$\pi_1(\lambda, t) = \begin{pmatrix} \lambda & 0 & t \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \pi_2(\lambda, t) = \begin{pmatrix} \lambda & 0 & t \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}.$$

Then Theorem 4 implies that the algebraic hulls of the $H_i = \pi_i(H)$-action in the direction perpendicular to the orbits is just $\mathrm{Ad}_G(H_i)|_{\mathfrak{g}/\mathfrak{h}_i}$ (up to subgroups of finite index); it is easy to show that $\mathrm{Ad}_G(H_1)|_{\mathfrak{g}/\mathfrak{h}_1}$ and $\mathrm{Ad}_G(H_2)|_{\mathfrak{g}/\mathfrak{h}_2}$ are not conjugate in $\mathrm{GL}(\mathfrak{g})$ (for instance $\mathrm{Ad}_G(H_2)|_{\mathfrak{g}/\mathfrak{h}_2}$ has a one-dimensional subspace of fixed vectors and $\mathrm{Ad}_G(H_1)|_{\mathfrak{g}/\mathfrak{h}_1}$ does not), showing that the H_1-action and the H_2-action cannot be smoothly orbit equivalent.

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