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Determinantal rings, by Winfried Bruns and Udo Vetter. Springer Lecture Notes 1327, Springer-Verlag, Berlin, Heidelberg, New York, 1988, vii+236 pp., $20.00. ISBN 3-540-19468-1

Determinantal varieties were considered for the first time in the nineteenth century in connection with the first and second fundamental theorem of invariant theory.

Let us consider a vector space F of dimension r over a field k. Let Z denote the space of $(m+n)$-tuples

$$(x, \xi) = (x_1, \dots, x_m, \xi_1, \dots, \xi_n)$$

of m vectors and n covectors (i.e., $x_i \in F$, $\xi_j \in F^*$). We consider the natural action of the group $\mathrm{GL}(r)$ of linear automorphisms of F on Z. Then the first fundamental theorem of invariant theory says that the ring of $\mathrm{GL}(r)$-invariant polynomial functions on X is generated by natural invariants $X_{ij}(x, \xi) = \xi_j(x_i)$ for $1 \le i \le m$, $1 \le j \le n$. The second fundamental theorem describes the ideal of relations between X_{ij}'s. It is generated by the $r+1$-order minors of the $m \times n$ matrix $X = (X_{ij})$.

It turned out later that the ideals generated by minors of a matrix appear naturally in algebraic geometry. For example, the singular locus of a variety is "naturally" defined by the vanishing of minors of the proper order of the Jacobian matrix.

Ideals of this type were studied systematically for the last thirty years from an algebraic point of view. Let B be a commutative ring, $X = (X_{ij})$ be the $m \times n$ matrix of indeterminates over B. Then the determinantal ring is the factor ring $R_t(X) = B[X]/I_t(X)$, where $B[X]$ is the ring of polynomials in indeterminates X_{ij} with coefficients in B, and $I_t(X)$ is the ideal in $B[X]$ generated by the minors of order t of the matrix X.

The investigation of such rings became one of the central topics in commutative algebra. This research retained its importance for invariant theory since the action of $\mathrm{GL}(r)$ described above is one of very few classical actions for which the ring of invariants can be described explicitly. It became a rule that theorems proved about rings of invariants were first proved for determinantal varieties.

The first class of determinantal ideals for which deep results were proved was the maximal minors, i.e., the ideals $I_t(X)$ for $t = \min(m, n)$. Eagon and Northcott in [E-N] constructed the explicit free resolution of the ideals $I_t(X)$ in this case. This construction allowed one to conclude that if B is a field, then the ideal $I_t(X)$ is perfect; i.e., its codimension equals the homological dimension of $R_t(X)$ as a $B[X]$-module. Both numbers equal $m - n + 1$, if we assume that $m \geq n$. The class of perfect ideals is an important one since if I is an ideal in a polynomial ring, then I is perfect if and only if T/I is Cohen–Macaulay.

It took much longer to prove that if B is a field then all determinantal ideals are perfect. This was achieved by Hochster and Eagon in [H-E]. Their method consisted of constructing a big lattice of ideals containing the determinantal varieties and proving by induction the perfection of all ideals in the lattice. The construction of the lattice was suggested by invariant theory. This theorem was one of the clues for Hochster and Roberts in their proof of Cohen–Macaulayness of all rings of invariants of a reductive group acting on a nonsingular variety.

The new period in the development of the theory started with the attempts to develop characteristic free invariant theory, and to construct a finite free resolution of the rings $R_t(X)$ for general t. It turned out that the representation theory of the general linear group is a powerful tool that can be used. In the fundamental paper [DC-E-P 1] the authors used representation theory to find the primary decompositions of powers of determinantal varieties over a field of characteristic zero. The same authors also developed in [DC-E-P 2] a new method of proving perfectness of determinantal varieties—the method of algebras with straightening law.

It consists of deforming the determinantal ideal to the associated ideal generated by monomials (to get this ideal we can pick the leading monomial in lexicographic order in each minor) and thus reducing the proof of perfectness of a determinantal ideal to the perfectness of the ideal generated by monomials. For the ideals generated by monomials the criteria of perfectness were already worked out by Reisner and Stanley.

This method is very important since it is applicable to a larger class of varieties: Schubert varieties, varieties of complexes, etc. It also allows one to describe the Hilbert function of the rings $R_t(X)$.

At the same time Lascoux used representation theory of the general linear group to construct the minimal free resolution of the rings $R_t(X)$ over a field of characteristic zero. He used a geometric method developed by Kempf based on the use of higher direct images of sheaves and Bott's theorem.

The methods developed for determinantal varieties can be applied to other important varieties (Schubert varieties in various homogeneous spaces, determinantal varieties for symmetric and antisymmetric matrices, conjugacy classes in classical Lie algebras, complete symmetric varieties of DeConcini and Procesi, etc.). One can use them to find explicit descriptions of coordinate rings and defining equations of such varieties as well as to prove geometric properties of them (Cohen–Macaulayness, rational singularities, etc.). The investigation of these special varieties evolved into a new field of mathematics lying on the crossroads between commutative algebra, representation theory, invariant theory, and combinatorics.

There are important unsolved problems for determinantal varieties themselves, for example, the construction of minimal free resolutions over fields of positive characteristic.

The results discussed above are the subject of the book under review. The authors restrict themselves to determinantal varieties and Schubert varieties in Grassmannians. The material concerning other varieties (determinantal ideals of symmetric and antisymmetric matrices) is referred to in the comments at the end of each chapter. This seems to be a good choice, since otherwise the amount of material would be too big. The base of the authors' approach is the technique of algebras with straightening law.

Chapter 1 is introductory. Determinantal ideals and Schubert varieties are introduced. In Chapter 2 the ideals of maximal minors are treated. The basic tool is the Eagon–Northcott resolution. The exactness of this resolution is proved via the Peskine–Szpiro acyclicity lemma. The formula for the codimension of the ideals $I_t(X)$ is also proved.

Chapter 3 is devoted to generically perfect ideals. This is a very useful notion which allows one to transfer many results on the ideals $I_t(X)$ to the results on the ideals generated by the t order minors of a matrix over some commutative ring, as long as codimension is preserved.

In Chapters 4 and 5 algebras with straightening law are introduced. In Chapter 4 the basic straightening law on the ring $B[X]$ is discussed. One also proves there that the coordinate rings of Grassmannians have a natural straightening law. These two are connected since the space of $m \times n$ matrices is the open subset in the Grassmannian of m—subspaces in $(m + n)$—space. In Chapter 5 the theory of algebras with straightening law on arbitrary partially ordered sets is developed. The notion of wonderful partially ordered sets is introduced and it is proved that algebras with straightening law on a wonderful partially ordered set over a Cohen–Macaulay ring of coefficients are also Cohen–Macaulay.

In Chapter 6 the authors prove that the determinantal and Schubert varieties are normal, and they calculate their singular locus.

Chapter 7 is devoted to invariant theory. The coordinate rings of determinantal and Schubert varieties are constructed as rings of invariants. The generic points of those varieties are constructed.

Chapter 8 deals with the class group and canonical divisors of the rings $R_t(X)$.

In Chapters 9 through 11 the basic results of the paper [DC-E-P 1] are proved, and some of them improved upon. Chapter 9 deals with various graded rings associated to determinantal rings. In Chapter 10 one calculates the symbolic powers of the ideals $I_t(X)$. In Chapter 11 representation theory is discussed. One constructs the irreducible representations of the general linear group using the straightening law and uses them to calculate primary decompositions of products of determinantal ideals. The method of U-invariants is also discussed.

Chapter 12 introduces the Hochster–Eagon original proof of perfection of the ideals $I_t(X)$ through principal radical systems of ideals.

The remaining three chapters are devoted to the study of linear algebra over the rings $R_t(X)$. The homological properties of generic modules are discussed, the module of Kahler differentials is constructed, and the rigidity of determinantal rings is proved.

Chapter 16 is the Appendix. It contains various commutative algebra results used throughout the text.

The book is written very carefully. The formulations and proofs of the results are very precise. Each chapter concludes with a brief section containing a description of related results not covered in the book and providing references. The subject index and index of notations are complete and easy to use. The book should become a good reference for determinantal varieties.

However certain criticism has to be expressed concerning the authors' approach to the subject. As indicated above, the field of determinantal varieties borders on geometry, commutative algebra, representation theory, and combinatorics. It seems that a book on the subject should reflect the balance between these components. The authors' approach is very algebraic. Geometry and representation theory are eliminated wherever possible, sometimes in an artificial way. An example of this bias is seen in the treatment of symbolic powers and primary decompositions. It seems to the reviewer it would be much better to first introduce irreducible representations and then prove the results of Chapters 9 and 10, simply because the reader can understand in this way much more easily why the results have to be true. Also in various places results from commutative algebra and combinatorics are referred to as "elementary," as opposed to representation theory results. The purely algebraic treatment also accounts for a certain dryness of the text.

The bibliography is pretty complete. It contains many papers on related subjects not used directly in the text. The only omission the reviewer noticed is the absence of works of Lakshmibai and Seshadri on standard bases of Schubert varieties in homogeneous spaces. This work, it seems, should be discussed briefly at the end of various chapters. Some comments about Kempf's method of higher direct images could also be included.

Overall, the book under review is a good, solid text which can serve as a reference to the subject and should be accessible to graduate students interested in determinantal varieties.

References

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