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*Algebras, Lattices, Varieties*, vol. 1, by Ralph N. McKenzie, George F. McNulty, and Walter F. Taylor. Wadsworth & Brooks/Cole, Monterey, California, 1987, 361 pp., $44.95. ISBN 0-534-07651-3

Lest the title leave any uncertainty, Volume 1 initiates a comprehensive four-part overview of *universal algebra* as the subject is understood today. It concerns properties of *algebras* that are, by and large, independent of their particular operational type. Special algebras, such as *lattices*, are dealt with from the point of view of their role in the study of universal algebras (nonempty sets augmented with an arbitrary system of finitary operations). *Varieties*, or equational classes of algebras, arise as one of the central themes in universal algebra. This volume presents a thorough and exquisitely executed account of the foundations of universal algebra together with a fine exposition of several sample results that illustrate the depth and the beauty of the subject.

The sheer quantity of new work published in universal algebra makes a strong case for the need for such a series. The Mathematical Reviews' Mathematics Subject Classification encompasses most of the universal algebra in two categories: 06XXX Order, Lattices, Ordered Algebraic Structures; and 08XXX General Mathematical Systems. But the 1970 version, which offered the single letter sections 06AXX and 08AXX, quickly proved to be a poor reflection of the explosion of research that was erupting. Grätzer [3] estimated that about a thousand publications in universal algebra appeared between 1968 and 1979, and it seems likely that an equal

number have appeared since. Responding to such areas of in-
creased activity, the 1980 Classification significantly reorganized
and expanded ten of its sixty categories. Among these ten were
both 06XXX, expanded to 06AXX, 06BXX, 06CXX, 06DXX,
06EXX, and 06FXX; and 08XXX, expanded to 08AXX, 08BXX
and 08CXX.

So what is the impetus for all of this work? Algebra, as a tool
for the study of mathematical structures apart from the classical
number systems, was originated and established in the nineteenth
century. Groups, rings, fields, Boolean algebras, vector spaces,
semigroups, and lattices arose as useful abstractions to model a
spectrum of mathematical objects. More focused study of the
properties and structure of each of these classes led to a deeper
understanding of the objects they modeled. Much of this work
was assembled in A. N. Whitehead's 1898 *A Treatise on Universal
Algebra* [4] where, for the first time, a unified approach to these
varied algebras was proposed. Many fundamental notions proved
to be totally independent of the choice of a particular operational
type of algebras: subalgebras, quotients, direct products, and au-
tomorphisms, as a beginning. Likewise, the proofs of many results
from special algebra were shown to rely on hypotheses expressible
in terms of only these notions. The experience of the past sixty
years has repeatedly shown that many new and deep results about
algebras depend on conditions which have nothing to do with a
special operational type. A new dimension of insight arises when
we turn from the historical classifications of algebras as groups,
modules, monoids, etc., to investigate instead algebras generating
congruence distributive, modular or permutable varieties, locally
finite algebras, algebras with decidable first-order theories, with
solvable word problems or perhaps with very large clones. This
fact is the central driving force behind universal algebra.

As the body of knowledge in this field has grown in breadth
and depth, the task of digesting it into textbook form has become
increasingly formidable. Perhaps G. Grätzer took the last oppor-
tunity to become a renaissance man of universal algebra when,
in 1968, he published a comprehensive account of the knowledge
of the day [2]. His second edition [3], appearing in 1979, made
no attempt to repeat this feat. Rather, it stood on its reputation
as a sound introduction to the foundations of the discipline and
added, as separate appendices by different authors, accounts of

subsequent work in five major areas. In 1981 the book by Burris
and Sankappanavar [1] appeared. This relatively compact, well-
composed, and very successful text again made no pretense of be-
ing exhaustive. It gave a thorough review of basic principles and
then efficiently led the reader to an exposition of several chosen
highlights, particularly those at the junction of universal algebra
and mathematical logic. But, in the meantime, the absence of a
reasonably comprehensive account of the last twenty years' devel-
opments has left for the novice insufficient access to a reservoir of
folklore which is viewed as well known by the experts.

It is this latter need that the present series promises to fill, and
Volume 1 is a major step toward doing so. The exposition is
clear, clean, and very readable. Frequent references, both forward
and backward, help the reader orient the material in a broader
context. Citations to earlier (but easily forgotten) definitions are
often provided. References to relevant historical motivation are
tightly interwoven with the development of mathematical content.
Many sections end with references to either (an apparently well-
formulated) Volume 2 or (more vaguely) to "subsequent volumes."
Problems and exercises at the end of each section are ample and
substantive.

The topics presented in this text seem to fall into three cat-
egories: development of the standard tools used to understand
algebra, presentation of examples to which these tools can be ap-
plied, and extraction of major results which demonstrate the power
and strength of these tools. I will address each of these categories
separately.

Chapters 1, 2, and 4, which form the bulk of the book, are
primarily dedicated to a systematic *development of the fundamental
concepts and working tools of universal algebra*. Chapter 1 provides
the notational and conceptual framework required for subsequent
work.

Chapter 2 gives both a logical and historical account of the cen-
tral and ubiquitous role of lattices in the study of algebra. Here
we see how the emergence of lattices of congruences, subalgebras,
clones, and varieties in universal algebra leads to a concentrated
study of lattice theory itself. Many classical results, such as the
Jordan–Hölder theorem, are seen to arise from the modularity of
the underlying congruence lattices, thus having led Dedekind, Ore,
and Birkhoff to initiate a careful study of modular lattices. Simi-

larly it is the distributivity of their congruence lattices, apart from
any other properties associated with their operational type, that ac-
counts for many of the special properties of lattices. (For example,
Baker's proof that a finite algebra in a congruence distributive va-
riety has a finite equational basis for its identities was suggested by
McKenzie's earlier proof for the special case of lattices.) Distribu-
tive lattices manifest very strong properties since they turn out to
be exactly the lattices representable by set union and intersection.

Chapter 4, the real core of the text, utilizes the previously es-
tablished foundation to give a smooth and coherent exposition of
a large number of major fundamental results. Clones are care-
fully treated for future use since the universal algebraist is often
not concerned with any particular choice of generating operations.
Next the authors treat several topics which were first done in the
context of various special algebras, but which rely in no way on
their special properties: the isomorphism theorems, congruence
generation, and subdirect and direct representation. Since most
lattices that arise from algebras are algebraic lattices, this class of
lattices also earns a systematic treatment. Congruence distributiv-
ity and ($n$-)permutability are known to be responsible for many of
the important properties that have been found in special algebras.
Since these topics are presented before the section on free alge-
bras, the otherwise easy proofs of their Maltsev characterizations
are not yet accessible. But this problem is remedied when, later,
they find their places in the section on interpretation of varieties.
Class operators, free algebras, and Birkhoff's theorem come next.
Chapter 4 concludes with an introduction to the commutator, fo-
cusing on Abelian algebras, and a promise to treat this important
tool in full in Volume 3.

*Examples and illustrations* of these concepts occur throughout
the book in the form of numbered examples and exercises. They
demonstrate the richness of applicability of the ideas and provide
short introductions to a wide selection of subsidiary topics. Chap-
ter 1 and (most of) Chapter 3 give further concrete examples from
which to draw later intuitions. As we see from Chapter 3, most
of the complexity of the subject can already be found in binary
algebras.

Finally, the authors have not failed to exposit a selection of
*major results* built with the tools they have developed. These re-
sults are sprinkled through Chapters 2 and 4 in the form of several

classical decomposition and representation theorems for distributive lattices, modular lattices, and projective geometries. But the primary harvest of terminal results occurs in the fifth chapter on unique factorizations of direct products into indecomposable factors. Here the authors have chosen a special advanced topic which has historically drawn people to universal algebra and lattice theory, and which itself builds on many components of the previous work. Unique factorization of finite sets (positive integers) into primes dates back to the early Greeks. Classical algebra abounds with unique factorization theorems: finite Abelian groups (Kronecker); groups, rings, and modules with chain conditions (Remak, Krull, Schmidt); finite-dimensional linear spaces (Wedderburn); projective geometries of finite height (Jónsson); and finite Boolean algebras. The fascination of this subject arises from several startling theorems which assure unique factorization under surprisingly weak hypotheses. Their proofs typically rely on our ability to recognize a decomposition by its projection congruences in the congruence lattice. Examples of such weak hypotheses are having permuting congruences, a finite congruence lattice, and a one-element subuniverse (Birkhoff, Ore); having modular congruences and being finite with a one-element subuniverse (Jónsson), and being finite with a one-element subuniverse that is a two-sided identity under some binary operation (Jónsson and Tarski). An unexpected result with a deceptively easy but most unconventional proof is that every finite algebra has at most one $k$th root for every $k$ (Lovász). Weak as these hypotheses are, the authors exhibit a multitude of examples to show how unique factorization fails under almost any relaxation of any of them.

The organization and choice of topics lend the book to a selection of different uses: an introductory text for the beginner, an advanced text for the intermediate, or a reference book for the expert. A good beginning graduate course might cover Chapters 1 and 2, and then include topics selected from Chapter 3 or later chapters. For a more advanced course, or for the reader having some working familiarity with the subject, a quick review of the first part of the book would be sufficient to begin Chapter 4. The fifth chapter offers an excellent opportunity to pursue one possible avenue into more advanced topics.

Finally, I would say that Volume 1 is a timely, important, and well-constructed contribution to mathematics, and I would highly

recommend it to anyone at all interested in universal algebra. For my own part, I will be anxiously awaiting the appearance of Volumes 2, 3, and 4.

## References

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Errata to the Review (Bull. Amer. Math. Soc., April 1989, pp. 252–256, by Roger Howe) of *Groups and Geometric Analysis* by Sigurdur Helgason.

In the discussion on pages 255–256 of this review, the symbol ♮, which in the book signifies $K$-bi-invariance, was omitted everywhere it should have occurred, with considerable loss of meaning. The places where ♮ should have appeared are as follows:

| | | |
|---|---|---|
| page 255 | line 28 | After $\mathscr{E}'(G)$ |
| page 255 | line 31 | After the second "and" |
| page 255 | line 33 | After $\mathscr{E}'(G)$ |
| page 255 | line 35 | After $\mathscr{D}(G)$ |
| page 255 | line 37 | After $\mathscr{E}'(G)$ (twice) |
| page 255 | line 43 | After $\mathscr{E}'(G)$ |
| page 255 | line 47 | After $\mathscr{E}'(G)$ |
| page 256 | line 2 | After $\mathscr{E}'(G)$ |
| page 256 | line 4 | After $\mathscr{E}'(G)$ |

Also, in case any readers of the review were unsure as to whether the six properties of spherical functions, listed on page 255, are