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Lattices with Unique Complements, by V. N. Saliĭ. Translations of Mathematical Monographs, Vol. 69. Translated by G. A. Kandall and B. Silver. American Mathematical Society, Providence, RI, 1988, ix+113 pp., $55.00. ISBN 0-8218-4522-5

1. Introduction

It is a good idea to introduce a branch of mathematics by discussing one of its interesting topics. If the discussion is thorough, then it can also provide the specialist with an up-to-date research survey.

Introducing lattice theory via the theory of uniquely complemented (UC, for short) lattices is especially appropriate. UC lattices form a natural foundation for Boolean algebras, a field well known to all mathematicians. In fact, the fundamental problem of this field originates in Boolean algebras: Is every UC lattice Boolean?

This branch of lattice theory is well-motivated, surprisingly deep, and its techniques have found widespread application.

2. The story

While in natural sciences we usually start the history of a field with "The ancient Greeks already knew that ... ," in lattice theory "ancient history" refers roughly to the years 1900–1935. Pioneers of lattice theory introduced the lattice concept as a tool in the axiomatization of Boolean algebras: C. S. Peirce, E. Schröder, and especially, E. V. Huntington. Independently, R. Dedekind introduced lattices and modular lattices as abstractions of ideals of algebraic numbers (rings).

Some of these early results are interesting and nontrivial. For instance, in 1904, E. V. Huntington characterized Boolean algebras as complemented lattices in which the complementation is pseudocomplementation, that is, $a \wedge x = 0$ implies that $x \leq a'$.

This and some similar results led Huntington to the conjecture that a UC lattice is Boolean (that is, distributive).

In "The Dilworth Theorems. Selected Papers of Robert P. Dilworth" (edited by K. P. Bogart, R. Freese, and J. P. S. Kung, Birkhäuser Verlag, to appear), the article by M. E. Adams on UC lattices paints a detailed picture (with 51 references) of the development of this field. In the period up to 1945, many papers were published in which theorems of the following type were proved: UC and property P imply distributivity. Property P could be "finite" (G. Birkhoff and M. Ward), "complete, atomic, and dually atomic" (G. Birkhoff and M. Ward), "orthocomplemented" (G. Birkhoff and M. Ward), "modular" (G. Birkhoff and J. von Neumann), and so on.

It was, therefore, very surprising when R. P. Dilworth proved in 1945 that every lattice can be embedded in a UC lattice. The proof was about 30 pages long and complicated; it introduced a number of interesting new ideas that proved to be very fruitful. Adams' article includes a detailed and readable description of Dilworth's paper.

The research after 1945 split into two streams with different goals:

A. To find weaker and weaker properties P such that UC and property P imply distributivity. Property P could be "atomic" (T. Ogasawara and U. Sasaki), "sectionally complemented" (L. Beran), "algebraic" (V. N. Saliĭ), and so on. Perhaps, the best result is that of J. E. McLaughlin: An atomic lattice with unique comparable complements (UCC, for short) is modular.

B. To develop further the techniques introduced by Dilworth; apply the new techniques to obtain simpler proofs of Dilworth's Theorem and also find new applications. R. A. Dean and others studied lattices freely generated by a poset; this found many applications and generalizations (M. E. Adams, D. Kelly, I. Rival, J. I. Sorkin, R. Wille, and others). Dilworth's concept of "covers" was generalized to free products of lattices (B. Jónsson, H. Lakser, C. R. Platt, and the reviewer).

These developments led naturally to more general types of free products called $\mathscr{C}$-free products (C. C. Chen and the reviewer) and $\mathscr{R}$-free products (M. E. Adams and J. Sichler), which yielded a number of simpler proofs of Dilworth's Theorem and also a variety of new results. Here are some samples: Every bounded, at most uniquely complemented lattice has a 0- and 1-preserving embedding into a UC lattice (C. C. Chen and the reviewer). A free product of hopfian lattices need not be hopfian (J. Sichler and the reviewer). There are $2^{\aleph_0}$ lattice varieties in which Dilworth's Theorem holds (M. E. Adams and J. Sichler). Adams' article in

"The Dilworth Theorems" contains many more examples and all the necessary references.

3. The telling

The book under review is divided into three chapters.

Chapter One (along with the introductory sections of Chapter Two and Three) is an introduction to lattice theory specifically tailored to the subject matter. Sections 1.18 to 1.20 are the most difficult; they contain the proof of the equivalence of the Kuratowski-Zorn Lemma, the Axiom of Choice, and the Well-Ordering Principle. Outside of these sections, no attention is paid as to whether or not the Axiom of Choice is used. For example, the theorem in §1.11 fails in ZF, and so does Birkhoff's Theorem (any distributive lattice can be represented by sets; see 4.7).

Chapter Two deals with properties that with UC imply distributivity (topic 2.A, discussed above), and presents a proof of Dilworth's Theorem. The first three sections are introductory, but they already contain McLaughlin's Theorem in 1.3 (I wish somebody could come up with an intuitively clear proof!) and Stone's Theorem (any Boolean lattice can be represented by sets) in 2.6. It is odd that the author does not point out that Birkhoff's Theorem (1.4.7) and Stone's Theorem (2.2.6) trivially imply each other; two different proofs are presented.

Section 4 is a full discussion of properties that with UC imply distributivity. There are some easy but useful general observations one can make:

Observation 1. *Let* L *be a* UCC *lattice. If* $a < b$ *in* L, *then* $b \wedge a' > 0$.

Proof. Since $a \vee a' = 1$ and $a < b$, it follows that $b \vee a' = 1$. So $b \wedge a' > 0$, since $b \wedge a' = 0$ would imply that a' has two comparable complements, namely, a and b.

In the next observation, "$a \prec b$" denotes that "a is covered by b," that is, that $a < b$ and there is no element in between.

Observation 2. *Let* L *be a* UCC *lattice. If* $a \prec b$, *then* $c = b \wedge a'$ *is an atom, and it is a relative complement of* a *in* $[0, b]$.

Proof. $c = b \wedge a' > 0$ by Observation 1. Thus $a \prec b$ implies that c is a relative complement of a in $[0, b]$. Similarly, $d = a \vee b'$ is a relative complement of b in $[a, 1]$. Since L is a UCC lattice and $a \prec b$, any t with $t \leq c$ is a complement of d; it follows that c is an atom.

Observation 3. *If x and y are relative complements of a in $[0, b]$ and c is a relative complement of b in $[a, 1]$, then x and y are complements of c.*

Proof. $c \vee x = c \vee a \vee x = c \vee b = 1$, and, similarly, $c \wedge x = 0$.

These observations show that there is some connection between UC, UCC, modularity, and distributivity. Since a lattice is distributive iff it is at most uniquely relatively complemented, Observation 3 is easily applicable to proving distributivity (and, similarly, modularity). Observation 2 shows, for instance, that if a UCC lattice is atomic, then it is dually atomic, and that the covering relation is preserved at least under some joins and meets.

These observations should have been stated at the beginning of the chapter. Observation 1 is Lemma 1 of §4; it is proved directly many times in §1. The use of Observations 2 and 3 would have simplified some proofs and eliminated many repetitions (see, for instance, pp. 28, 29, 41, 43, 47, and 99).

In a UC lattice, call an element a "regular" iff

$$a \wedge x = 0 \text{ and } a \wedge y = 0 \text{ imply that } a \wedge (x \vee y) = 0.$$

This is a distributivity condition on a. Surprisingly, in a UC lattice, all atoms are regular (Lemma 3). Saliĭ's main result in this section is

Theorem. *A* UC *lattice is distributive iff each nonzero element contains a nonzero regular element.*

Section 5 reproduces M. E. Adams and J. Sichler's proof of Dilworth's Theorem in the smallest proper lattice variety to which it applies. It is regrettable that there is no mention of $\mathscr{C}$-free products, $\mathscr{R}$-free products, and the various applications of the techniques inspired by these investigations.

Chapter Three deals with complete UC lattices. This is a fairly long chapter considering that practically nothing is known about complete UC lattices. The first two sections are introductory; they contain a description of the the MacNeille completion of a lattice and Glivenko's Theorem (the MacNeille completion of a Boolean lattice is Boolean).

Section 3 contains an interesting new result. A "P-operation" o on the set A is a map of $P^*(A)$ (the set of nonempty subsets of A) into A. In a complete lattice, the complete join $\bigvee$ (and also $\bigwedge$) is an example of a P-operation. A subset B of A is a "P-subalgebra" (called "P-suboperative" in this book) of (A, o) if $o(X) \in B$ for all $X \subseteq B$, $X \neq \varnothing$.

Theorem. *Every complete lattice L can be represented as the lattice of all* P*-subalgebras of (A, o), where o is a* P*-operation on A. Moreover, we can assume that every* P*-subalgebra is generated by a singleton.*

Section 4 is a rather odd discussion of algebraic lattices. It starts with the Birkhoff-Frink theorem (characterizing subalgebra lattices as algebraic lattices) with a fairly long proof, even though the germs of a trivial proof can be found on p. 72. In §6 collectively complemented lattices (lattices that admit a homomorphism onto a UC lattice) are introduced, and a characterization of such lattices due to V. V. Pashenkov is presented.

Section 7 first deals with "orthogonal subsets" (any two distinct elements meet in 0) in a complete UC lattice. For a finite set A and a complete lattice L, an "L-relation" is a map $f : A^2 \to L$. There is a natural definition of the product of L-relations. An "L-transformation" f is an L-relation satisfying $f(x, y_1) \wedge f(x, y_2) = 0$ whenever $y_1 \neq y_2$. The rest of the section deals with conditions under which the set (or some subset) of transformations form a semigroup.

In the final section it is proved that in a complete UC lattice the regular elements form a complete sublattice. It is an easy corollary that a complete UC lattice is a direct product of a complete and atomic Boolean lattice and a complete UC lattice without atoms. The first two lemmas of this section imply that in a UC lattice an element is regular iff it is neutral iff it is central. (An element a of a lattice L is called "neutral" iff a, x, and y generate a distributive sublattice, for any x, $y \in L$; an element a of a lattice L is called "central" iff a has a complement a' and $x \to \langle x \wedge a, x \wedge a' \rangle$ is an isomorphism between L and $(a] \times (a']$.) This should have been proved in Chapter Two; it may have helped with some proofs and it would have shown that regularity is just a way of introducing the old concept of neutrality for UC lattices. Some of the results of this section are simply restatements of known results about neutral elements.

4. The translation

The book was translated by G. A. Kandall and edited by B. Silver. They should be commended for a book that is easy to read (linguistically). Unfortunately, some other issues were not considered. The Russian edition is meant for students who are familiar with basic algebraic concepts, including universal algebras, congruences, varieties, identities, quasi-identities, Birkhoff's Subdirect Product Representation Theorem, etc. A three- or four-page introduction could have summarized the necessary concepts and results, or at least could have given a guidance as to what could be found where. For instance, the Preface suggests that for a general introduction to lattices and Boolean algebras the reader should turn to two books not available in English.

The problems with terminology start with the first sentence of Chapter One. Literally translated, Saliĭ calls posets "p-sets." The translator changes this to "o-sets," which has a connotation of (fully) ordered sets. Why not use "posets" the standard short form of partially ordered sets? Prime intervals in Chapter One are introduced as "simple intervals;" fortunately, in the other two chapters they are called "prime." The "Substitution Property" is called "stability" without introducing the concept (and on p. 67 it means that a set is closed under a P-operation), and the "diamond" (the lattice M_3) is called the "rhombus." The "signature" of an algebra is not introduced (it is like the "type," but includes the names of the operations). "Quasi-identities" are used but not defined, and "sectionally complemented" lattices are called "initially complemented."

The number of typographical errors is unusually high. Some of these do not interfere with the mathematical contents (for instance, no space before and after "&" in many formulas), but some make reading the book difficult. For instance, on p. 46, 1.3, the definition of the element "v" should be "$(x \wedge u) \vee (x \wedge y')$;" on p. 47 in Theorem 16, "$<$" should be "$>$"; on p. 56, the formula defining "O-modular" lattices should read "$x \leq z \,\&\, y \wedge z = 0 \Rightarrow (x \vee y) \wedge z = x$;" on p. 66, "If $X \subseteq A$ and $\mathscr{X} \subseteq \operatorname{Sup} A$, then (assuming, of course, that $X \neq \varnothing$ and $\mathscr{X} \neq \varnothing$)" should read "If $X \subseteq A$ and $\mathscr{X} \subseteq \operatorname{Sub} A$, then (assuming that $X \neq \varnothing$ or $\mathscr{X} \neq \varnothing$)." I counted 6 typographical errors in formulas on p. 32, and 11 on p. 33.

5. In conclusion

If I sound critical of the book, it is because I basically like it, and I am somewhat disappointed with the final result. Most of the faults could have been easily eliminated by an expert in lattice theory in a few days. And why are words misspelled in these days of spelling checkers? Why is a section heading abbreviated to make it mathematically meaningless (§5 of Chapter Two)?

More important than these faults is the enthusiasm of the author for the subject matter. Everybody I discussed this book with became quite enchanted with UC lattices. This topic is full of surprising and deep results, yet even some of the most fundamental problems are unresolved. To whet the reader's appetite, let me mention three:

1. Is there a complete nondistributive UC lattice?
2. Is the MacNeille completion of a UC lattice a UC lattice?
3. Is there a simple "natural" example of a nondistributive UC lattice (in geometry, combinatorics, or topology)?

The first two problems are due to R. P. Dilworth; note that the MacNeille completion of a distributive lattice is not necessarily a distributive lattice. The first explicit mention of the third problem seems to be in the reviewer's General Lattice Theory, but I am sure that experts in the field considered it earlier.

G. Grätzer
University of Manitoba

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Grassmannians and Gauss maps in piecewise-linear topology, by Norman Levitt. Lecture Notes in Mathematics, vol., Springer-Verlag, 1989, $21.10. ISBN 0-50756-6

Combinatorial geometry goes back at least to Euclid and the study of regular polyhedrons in Euclidean space, using both metric and linear properties. In the 18th and 19th centuries, Euler and Poincaré exploited purely combinatorial properties to define the Euler class, homology and Poincaré duality of general polyhedral surfaces; and Riemann exploited local metric properties of smooth