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A SHARP COUNTEREXAMPLE ON THE REGULARITY OF Φ-MINIMIZING HYPERSURFACES

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A standard problem in the calculus of variations seeks a hypersurface S of least area bounded by a given $(n-2)$-dimensional compact submanifold of $\mathbf{R}^n$. More generally, given any smooth norm Φ on $\mathbf{R}^n$, seek to minimize

$$\Phi(S) = \int_S \Phi(\mathbf{n}),$$

where $\mathbf{n}$ is the unit normal vector to S. Think of the integrand Φ as assigning a cost or energy to each direction. We assume that Φ is *elliptic* (*uniformly* convex), the standard hypothesis for regularity.

Geometric measure theory (cf. [M, Chapters 5, 8], [F 1, 5.1.6, 5.4.15]) guarantees the existence of a (possibly singular) Φ-minimizing hypersurface with given boundary. For the case of area ($\Phi(\mathbf{n}) = 1$), area-minimizing hypersurfaces are regular embedded manifolds up through $\mathbf{R}^7$, but sometimes have singularities in $\mathbf{R}^8$ and above. For general elliptic Φ, a result of Almgren, Schoen, and Simon [Alm S S, Theorem II.7] guarantees regularity up through $\mathbf{R}^3$, but there were no examples of singularities below $\mathbf{R}^8$. We establish the sharpness of the Almgren–Schoen–Simon regularity result by giving a singular Φ-minimizing hypersurface in $\mathbf{R}^4$.

The surface is the cone C over the Clifford torus $S^1 \times S^1 \subset \mathbf{R}^2 \times \mathbf{R}^2$:

$$C = \{(x, y) \in \mathbf{R}^2 \times \mathbf{R}^2 : |x| = |y| \leq 1\}.$$

The norm Φ depends smoothly on $\theta = \tan^{-1}(|y|/|x|)$ alone, so that we may view Φ as a norm on $\mathbf{R}^2$. The unit Φ-ball is pictured in Figure 1. Any smooth, symmetric, uniformly convex approximation of the square will do. Note that Φ is smaller (say 1) on

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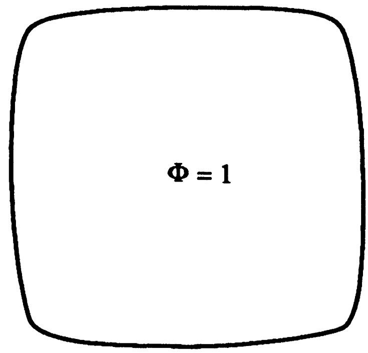

FIGURE 1. The unit Φ-ball. The smallness of Φ in the diagonal directions helps to make the cone C Φ-minimizing.

the normals to diagonal directions, which occur in the cone C, so that $\int_C \Phi(\mathbf{n})$ is relatively small.

The unit ball of the norm dual to Φ, called the *Wulff crystal* $W(\Phi)$, is pictured in Figure 2. The Wulff crystal itself solves an important problem: its boundary surface S minimizes $\Phi(S)$ for fixed volume enclosed (cf. [T, §1]). In nature $\Phi(S)$ represents the surface energy of a crystal, and the Wulff crystal $W(\Phi)$ gives the shape which a fixed volume of material assumes to minimize surface energy. The Wulff crystal of our norm Φ resembles a pivalic acid crystal (see Figure 3).

The Proof. The proof that the cone C over $S^1 \times S^1 \subset \mathbf{R}^4$ is Φ-minimizing employs the "method of calibrations" (cf. [HL, Introduction]). One must produce a closed differential 3-form or "calibration" φ such that for any point p and unit 3-plane ξ, with unit normal $*\xi$,

$$\langle \xi, \varphi(p) \rangle \leq \Phi(*\xi), \tag{1}$$

with equality whenever ξ is the oriented unit tangent to C at p. Then if S is any other surface with the same boundary,

$$\Phi(C) = \int_C \varphi = \int_S \varphi \leq \Phi(S),$$

so that C is Φ-minimizing.

Finding a calibration φ remains an art, not a science. Our calibration is

$$\varphi = (\sin^2 2\theta \, dr + \sin 4\theta \, r d\theta) \wedge d\theta_1 \wedge d\theta_2,$$

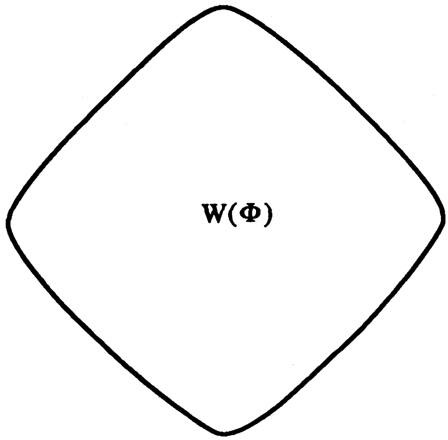

FIGURE 2. The Wulff crystal $W(\Phi)$ may be defined as the unit ball for the norm dual to Φ. For fixed volume, $W(\Phi)$ has the least surface energy measured by Φ.

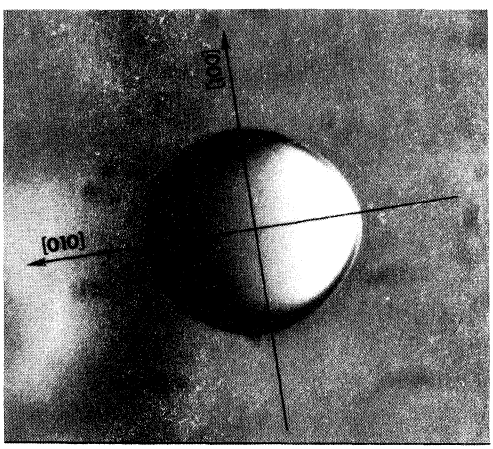

FIGURE 3. The pivalic acid crystal resembles the Wulff crystal $W(\Phi)$ of Φ [GS].

where $r^2 = |x|^2+|y|^2$, $\theta = \tan^{-1}(|y|/|x|)$, $\theta_1 = \arg x$, $\theta_2 = \arg y$.

It resembles the 7-form of H. Federer's proof [F2, §6.3] after H. B. Lawson [L, §5] that the cone over $\mathbf{S}^3 \times \mathbf{S}^3$ is area-minimizing. At any point in our cone C, $\theta = \pi/4$, and $\varphi(\pi/4) = dr \wedge d\theta_1 \wedge d\theta_2$ is precisely dual to each unit tangent ξ_0 to C. Hence

$$\langle \xi, \varphi(p) \rangle \leq 1 \leq \Phi(*\xi),$$

with equality whenever $\xi = \xi_0$. Thus (1) holds at points $p \in C$. Unfortunately, for $p \notin C$ (for example $\theta = \pi/8$), the $\sin 4\theta$ term, which is necessary to make φ closed, tends to make φ big. In order for (1) to hold, the largeness of $\varphi(\pi/8)$ must be somehow compensated for by the largeness of $\Phi(\pi/8)$.

Establishing the estimate (1) at all points almost always is a main difficulty.

For the case of area, the right-hand side is 1, and the estimate becomes $|\varphi(p)| \leq 1$, independent of ξ. For a general integrand Φ, the estimate involves both p and ξ. This difficulty explains why calibrations have not been applied specifically to integrands other than area before.

We handle this difficulty with a lemma that associates with φ the function on unit vectors

$$G(w) = \sup\{|\varphi(p)| : w \text{ is the oriented unit normal to the } (n-1)\text{-plane dual to } \varphi(p)\}.$$

The lemma says that the desired estimate (1) holds if the graph of G lies inside the Wulff crystal $W(\Phi)$, thus reducing the required estimate to a single parameter.

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