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AN EXTENSION OF CASSON'S INVARIANT TO RATIONAL HOMOLOGY SPHERES

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In 1985, Andrew Casson defined an invariant $\lambda(M)$ of an oriented integral homology 3-sphere M [C, AM]. This invariant can be thought of as counting the number of conjugacy classes of non-trivial representations $\pi_1(M) \rightarrow SU(2)$, in the sense that the Lefschetz number of a map counts the number of fixed points. Casson proved the following three properties of λ .

- (i) If $\pi_1(M) = 1$, then $\lambda(M) = 0$.
- (ii) Let N be the complement of a knot in a homology sphere and let $N_{1/n}$ denote N Dehn surgered along one meridian and n longitudes (see below for terminology). Then

$$\lambda(N_{1/n}) = \lambda(N) + n\Delta_N''(1),$$

where $\Delta_N''(t)$ is the second derivative of the Alexander polynomial of N .

- (iii) $4\lambda(M)$ is congruent modulo 16 to the μ -invariant (see below) of M .

This paper describes an extension of Casson's methods to the case where M is a rational homology 3-sphere, including generalizations of (ii) and (iii). (This extension is different from the one given in [BN].) In addition, an alternate definition of λ , using the generalized Dehn surgery formula, is given (Theorem 1).

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Let M be a rational homology 3-sphere ($H_*(M; \mathbf{Q}) \cong H_*(S^3; \mathbf{Q})$), and let (W_1, W_2, F) be a Heegaard splitting of M (i.e., M is the union of handlebodies W_1 and W_2 , and $\partial W_1 = \partial W_2 = W_1 \cap W_2 = F$). Let g be the genus of F . Let

$$R = \text{hom}(\pi_1(F), SU(2))/SU(2)$$

$$Q_j = \text{hom}(\pi_1(W_j), SU(2))/SU(2),$$

where $SU(2)$ acts via conjugation. R and Q_j are smooth, compact, locally conelike stratified spaces of dimensions $6g - 6$ and $3g - 3$, respectively. It is, therefore, tempting to try to define an intersection number $\langle Q_1, Q_2 \rangle$ of Q_1 and Q_2 in R .

Denote the trivial representation in R by 1. If M is an integral homology sphere ($H_*(M; \mathbf{Z}) \cong H_*(S^3; \mathbf{Z})$), then $Q_1 \cap Q_2$ does not contain any singular points of R , apart from 1, which is isolated. Hence Q_1 can be isotoped into general position with respect to Q_2 via an isotopy supported away from the singularities of R . Define $\langle Q_1, Q_2 \rangle$ to be the signed (finite) number of irreducible representations in $Q_1 \cap Q_2$ after this isotopy. It is easy to see that this is independent of the choice of isotopy. Up to sign (and a factor of $\frac{1}{2}$), $\langle Q_1, Q_2 \rangle$ is Casson's definition of $\lambda(M)$. It is easy to show, using the Reidemeister–Singer theorem, that $\lambda(M)$ does not depend on the choice of Heegaard splitting of M .

If M is a rational homology sphere (RHS), $Q_1 \cap Q_2$ meets the singularities of R , and things are more complicated. Before giving the definition of $\langle Q_1, Q_2 \rangle$ in this case, it will be necessary to discuss the singularities of R in more detail. (The basic reference for this is [Gol].)

R has a stratification

$$R \supset S \supset P,$$

where S consists of representations with Abelian image and is diffeomorphic to a $2g$ -torus modulo a \mathbf{Z}_2 action, and P consists of representations into \mathbf{Z}_2 (the center of $SU(2)$). Q_j has a similar stratification

$$Q_j \supset Q_j \cap S \supset Q_j \cap P.$$

Let $R_- = R \setminus S$, $S_- = S \setminus P$. Let ν be the Zariski normal bundle of S_- in R and let η_j be the Zariski normal bundle of $Q_j \cap S_-$ in Q_j .

R has a natural symplectic structure with respect to which Q_j is Lagrangian. This symplectic structure induces a symplectic structure of ν with respect to which η_j is a Lagrangian subbundle

(defined over $Q_j \cap S_-$). The first Chern class of ν , $c_1(\nu)$, is represented by the symplectic form ω restricted to S_- .

Since M is a RHS, Q_1 and Q_2 are in general position near P , and $Q_1 \cap S$ and $Q_2 \cap S$ are in general position inside S . It follows that Q_1 can be put into general position with respect to Q_2 via a symplectic isotopy of R which is fixed near P and on S . After this isotopy, $Q_1 \cap Q_2$ consists of a finite number of points. For $p \in Q_1 \cap Q_2 \cap S \setminus \{1\}$, we wish to define a number $I(p)$ so that

$$(1) \quad \langle Q_1, Q_2 \rangle = \sum_{p \in Q_1 \cap Q_2 \cap R_-} \text{sign}(p) + \sum_{p \in Q_1 \cap Q_2 \cap S \setminus \{1\}} I(p)$$

does not depend on the choice of general positioning isotopy.

Let $p \in Q_1 \cap Q_2 \cap S_-$. Let L_j be the fiber of η_j at p . If, at some time during a generic isotopy of Q_1 , L_1 is not transverse to L_2 , then an irreducible intersection of Q_1 and Q_2 (i.e. a point of $Q_1 \cap Q_2 \cap R_-$) is created or destroyed near p . Hence $I(p)$ must be defined so that it changes by ± 1 when this occurs.

Let α_j be an arc from 1 to p in $Q_j \cap S$. Let γ be the loop $\alpha_1 * (-\alpha_2)$. Choose a complex structure on F . This induces an almost Kähler structure on ν with respect to which η_j is a totally real subbundle. η_j gives rise to a section $\det(\eta_j)$ of the determinant line bundle $\det(\nu)$ of ν over γ . Patching $\det(\eta_1)$ and $\det(\eta_2)$ together in a canonical fashion over 1 and p determines a (homotopy class of) trivialization Φ of $\det(\nu)|_\gamma$. The patching procedure is such that when L_1 passes through a Lagrangian not transverse to L_2 , Φ changes by ± 1 .

Let $E \subset S$ be a surface bounded by γ . Let $c_1(\det(\nu)|_E, \Phi)$ be the relative first Chern class. Since $c_1(\nu) \neq 0$, this depends on the choice of E . But since $c_1(\nu)$ is represented by ω ,

$$I(p) = c_1(\det(\nu)|_E, \Phi) - \int_E \omega$$

is independent of the choice of E and also has the properties necessary to make (1) above isotopy invariant.

(Actually, the above constructions must be carried out equivariantly in the double cover of S . This is necessary in order for $\det(\nu)$ to extend over P and for η_j to be an oriented bundle. (The orientation determines the sign of $\det(\eta_j)$.) Also, there is not a unique candidate for the canonical patching procedure, but rather two equally canonical candidates, so Φ should be thought of as the average of the two resulting trivializations.)

Finally, define

$$(2) \quad \lambda(M) = \frac{\langle Q_1, Q_2 \rangle}{|H_1(M; \mathbf{Z})|}.$$

(Dividing by $|H_1(M; \mathbf{Z})|$ makes λ additive under connected sums and leads to a nicer Dehn surgery formula.) It is not hard to show that (2) does not depend on the choice of Heegaard splitting for M .

It is possible to prove a Dehn surgery formula for λ which generalizes (ii) above. Before stating it, it will be necessary to give a few definitions.

Let K be a knot in a RHS. Let N be the complement of a tubular neighborhood of K . Let Δ_N be the Alexander polynomial of N . Normalize Δ_N so that $\Delta_N(1) = 1$ and $\Delta_N(t^{-1}) = \Delta_N(t)$. Let $\Delta_N''(1)$ denote the second derivative of $\Delta_N(t)$ evaluated at $t = 1$.

Let $l \in H_1(\partial N; \mathbf{Z})$ be a longitude of N , that is, a generator of $\ker(H_1(\partial N; \mathbf{Z}) \rightarrow H_1(N; \mathbf{Z}))$. Let $\langle \cdot, \cdot \rangle$ denote the intersection pairing on $H_1(\partial N; \mathbf{Z})$. (The orientation of ∂N is induced from that of N via the “inward normal last” convention.) Let $a, b \in H_1(\partial N; \mathbf{Z})$ be primitive homology classes such that $\langle a, l \rangle \neq 0, \langle b, l \rangle \neq 0$. Choose a basis x, y of $H_1(\partial N; \mathbf{Z})$ such that $\langle x, y \rangle = 1$ and $l = dy$ for some $d \in \mathbf{Z}$. Define

$$\begin{aligned} \tau(a, b; l) &= -s(\langle x, a \rangle, \langle y, a \rangle) + s(\langle x, b \rangle, \langle y, b \rangle) \\ &\quad + \frac{1}{12} \left(1 - \frac{1}{d^2} \right) \left(\frac{\langle x, a \rangle}{\langle y, a \rangle} - \frac{\langle x, b \rangle}{\langle y, b \rangle} \right), \end{aligned}$$

where $s(q, p)$ denotes the Dedekind sum

$$\begin{aligned} s(q, p) &= \sum_{k=1}^{|p|} \left(\left(\frac{k}{p} \right) \right) \left(\left(\frac{kq}{p} \right) \right) \\ ((x)) &= \begin{cases} 0, & x \in \mathbf{Z} \\ x - [x] - \frac{1}{2}, & \text{otherwise.} \end{cases} \end{aligned}$$

Note that $\tau(a, b; l)$ depends only on a, b, l , and $\langle \cdot, \cdot \rangle$, not on x, y , or the topology of N .

For a a primitive element of $H_1(\partial N; \mathbf{Z})$, let N_a denote the Dehn surgery of N along a . That is, $N_a = N \cup_f (D^2 \times S^1)$, where $f: \partial D^2 \times S^1 \rightarrow \partial N$ maps $\partial D^2 \times \{\theta\}$ to a curve representing a .

Proposition 1. *Let a , b , and l be as above. Then*

$$\lambda(N_a) = \lambda(N_b) + \tau(a, b; l) + \frac{\langle a, b \rangle}{\langle a, l \rangle \langle b, l \rangle} \Delta_N''(1).$$

The proof of Proposition 1 is long and technical, and relies heavily on the isotopy invariance of $\langle Q_1, Q_2 \rangle$.

One can actually take Proposition 1 as the definition of λ . In other words, one can give an elementary proof to the following theorem.

Theorem 1. *There is a unique \mathbf{Q} -valued invariant λ of rational homology spheres such that $\lambda(S^3) = 0$ and λ and λ satisfies Proposition 1.*

Note. Boyer and Lines [BL] have shown independently that the Dehn surgery formula of Proposition 1, applied to the case where N is a knot complement in an integral homology sphere, yields a well-defined invariant of homology lens spaces.

Sketch of proof. First one shows that any RHS M can be obtained from S^3 via a sequence of Dehn surgeries such that each intermediate manifold is a RHS. Call such a sequence a permissible surgery sequence. This sequence, together with Proposition 1 and the axiom that $\lambda(S^3) = 0$, determines $\lambda(M)$. This establishes the uniqueness of λ .

To establish existence, it must be shown that any two permissible surgery sequences resulting in the same RHS M yield the same value for $\lambda(M)$. Using standard techniques, it is possible to modify any permissible surgery sequence into an integral permissible surgery sequence (i.e., one such that each surgery curve intersects the corresponding meridian once). This modification does not affect the computation of $\lambda(M)$. Any integral permissible surgery sequence can be represented by an ordered framed link in S^3 . (The ordering is the order in which the components are to be surgered.)

Using Kirby's theorem [K], one can show that any two permissible ordered framed links representing the same RHS are related by a finite sequence of the following three moves: (1) Adding or subtracting an unknotted component of framing ± 1 , anywhere in the ordering; (2) "Sliding" a component over another component which precedes it in the ordering; (3) Transposing two components which are adjacent in the ordering (so long as this does not violate "permissibility").

Invariance of $\lambda(M)$ under moves (1) and (2) is easy to show. (Note that move (2) does not change (modulo isotopy) the surgery sequence represented by the ordered framed link.) Showing invariance under move (3) is more involved, but still elementary.

The following are easily derived from Proposition 1.

(A) Let M be a RHS and let $-M$ denote M with the opposite orientation. Then

$$\lambda(-M) = -\lambda(M).$$

(B) Let M_1 and M_2 be RHS's. Then

$$\lambda(M_1 \# M_2) = \lambda(M_1) + \lambda(M_2).$$

(C) Let $L_{p,q}$ denote the p, q lens space. Then

$$\lambda(L_{p,q}) = -s(q, p).$$

(It follows that $4|H_1(L_{p,q}; \mathbf{Z})|\lambda(L_{p,q})$ is equal to the signature defect of $L_{p,q}$ (see [HZ]).)

(D) $6|H_1(M; \mathbf{Z})|\lambda(M)$ is an integer for any M . (This uses the fact that $6ps(q, p)$ is always an integer.)

For M a \mathbf{Z}_2 -homology sphere, let $\mu(M)$ denote the signature, modulo 16, of a spin 4-manifold bounded by M . Proposition 1 and a mild generalization of Theorem 2 of [Gor] yield

Proposition 2. *For M a \mathbf{Z}_2 -homology sphere,*

$$4|H_1(M; \mathbf{Z})|^2 \lambda(M) \equiv \mu(M) \pmod{16}.$$

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