References

1. P. Beckmann, *The first* 140 *million*, Access to Energy **14** (1987), 1.
2. J. M. Borwein and P. B. Borwein, *Ramanujan and pi*, Scientific American **258** (1988), 112–117.

George E. Andrews
Pennsylvania State University

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 22, Number 1, January 1990

0273-0979/90 $1.00 + $.25 per page

Cordes' two-parameter spectral representation theory, by D. F. McGhee and R. H. Picard. Pitman Research Notes in Mathematics Series, Volume 177, Longman Scientific and Technical, Harlow, United Kingdom and New York, 1988, 114 pp., $41.95. ISBN 0-470-21084-2

Mathematical developments can be viewed as a river fed from numerous tributaries and giving rise to branching streams of vigorous activity, quiet meandering backwaters which may become brackish and stagnate or possibly return with renewed vigour to the main stream. Multiparameter spectral theory, of which McGhee and Picard's book deals with a particular but central aspect, is an example of such an analogy.

In order to discuss the central questions of multiparameter theory and its relation to other branches of classical and functional analysis it is necessary to formulate the general problem.

Suppose one has k separable Hilbert spaces H_r, $1 \leq r \leq k$ and a collection of linear operators T_r, V_{rs}, $1 \leq s \leq k$, defined on these spaces. One now forms the k linear combinations

$$W_r(\lambda) \equiv T_r + \sum_{s=1}^{k} \lambda_s V_{rs}, \qquad 1 \leq r \leq k \tag{1}$$

where $\lambda_s \in \mathbf{C}$ are scalars. The central question is then to determine the scalars $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k) \in \mathbf{C}^k$ such that all the linear operators $W_r(\boldsymbol{\lambda})$ have nonzero kernels. Briefly then, we have a multiparameter spectral problem invoking a plethora of questions thus generalising in a nontrivial manner one-parameter spectral theory. In particular it is essential to develop a framework in which to discuss the multiparameter generalisation of the classical representation theorem

$$A = \int \lambda \, dE_\lambda , \tag{2}$$

for a linear operator A defined on a Hilbert space H and whose spectral measure is E_λ.

In an ordinary differential equation setting multiparameter spectral theory can trace its origins almost as far back as the Sturm-Liouville theory itself. Indeed in the two-parameter case eigenvalue problems for $W_r(\boldsymbol{\lambda}) = 0$, $r = 1, 2$ were studied by D. Hilbert [9] in which eigenfunction expansion theorems were developed and by F. Klein [13] who established an oscillation theory. R. D. Carmichael [6] treated k-parameter matrix problems while A. J. Pell [15] considered pairs of Fredholm integral operators coupled by pairs of parameters. The motivating and driving force underlying multiparameter spectral theory emerges in the separation of variables technique for the solution of partial differential equations. In the most elementary case such as the oscillations of a rectangular membrane with fixed boundary one is led to two separate Sturm-Liouville problems which are separate not only in regard to their independent variables but also with respect to the spectral parameters (i.e. separation constants) as well. For a circular membrane there is mild coupling via spectral parameters. The full multiparameter situation occurs in the case of the elliptic membrane wherein the separated equations both contain the same two spectral parameters. The solutions underlying the oscillation problem in this case involve the Mathieu functions. In other canonical problems the separation of variables technique leads to a study of Lamé, ellipsoidal and spheroidal functions etc. which together with the Mathieu functions constitute the so called higher special functions of mathematical physics. Their importance to physics and particularly quantum mechanics in the early decades of this century attracted considerable attention by analysts and for a time the mainstream of multiparameter theory was somewhat neglected. Most of this early work on the higher special functions was brought together by A. Erdélyi in volume 3 of the Bateman manuscript project [8].

The first major return to the mainstream occurs between 1953 and 1955 when H. O. Cordes [7] developed an abstract Hilbert space setting for the method of separation of variables and established a spectral representation theorem for a class of two-parameter problems. Cordes' beautiful and fundamental ideas lie at the heart of recent work and were successfully used by him in application to a problem related to the stark effect of the hydrogen atom. In Cordes' work the following key assumptions are made:

(3)
(i) $V_{rs} \in B(H_r)$, (the set of bounded linear operators on the Hilbert space H_r), $1 \leq r, s \leq 2$.
(ii) $V_{11}, V_{12}, -V_{21}, V_{22} > 0$ and $|V_{r1}| + V_{r2} = \mathrm{Id}$, $1 \leq r \leq 2$, where Id is the identity.

Since one is interested in the study of the nonzero kernels of $W_r(\boldsymbol{\lambda})$, $r = 1, 2$ a "tensor product" construction is called for. Such a construction had previously been developed by F. J. Murray and J. von Neumann [14] in their work on rings of operators and consequently Cordes' developed his theory in a weighted version of the tensor product space

$$H^{\otimes} = H_1 \otimes H_2. \tag{4}$$

Perhaps because of the technical constructions and consequent lengthy and subtle proofs the importance of Cordes' work was not realised at the time. This, together with the vigorous and far reaching developments of classical spectral theory in the post war years by W. N Everitt, K. Kodaira, M. G. Krein, B. M. Levitan, M. A. Naimark, E. C. Titchmarsh and others seem to have diverted attention once more away from the mainstream. Nevertheless, new advances were being made by F. M. Arscott [1] in the understanding of the elusive ellipsoidal wave functions. Arscott [2] also returned to particular forms of the general case and formulated bi-orthogonality properties and formal expansion theorems.

On November 17, 1965 at the Iowa City meeting of the American Mathematical Society, F. V. Atkinson announced a broad programme of research introducing numerous seminal ideas for the future development of multiparameter spectral theory. Atkinson's survey paper [3] returned the subject back with renewed vigour to the mainstream to which it has held steady ever since. It laid the foundations for most of the advances which have taken place over the last two decades, but more than this it drew attention to the many connections with polynomials in commutative operator algebras including operator bundles. It outlined a basis for a functional calculus, eigenvalue notions and ideals, singular matrix pencils, chain complexes and much more. As yet few of these wide ranging algebraic concepts have been taken further and the main thrust of development has been in regard to multiparameter linear operator systems and their associated spectral properties not least of which has been the establishment of eigenfunction expansion theorems. Indeed the guiding light for this research is the observation that any spectral problem involving a single parameter will have nontrivial extensions involving several parameters.

In 1972 Atkinson [4] published volume 1 of what had been planned (or is planned?) as the first of a two-volume work on multiparameter eigenvalue problems. This work which concentrates on matrices and compact operators develops, in a systematic way, a number of the ideas initiated in his address of 1965.

A key idea exploited by Atkinson, but hinted at already by Cordes and the previous work on the higher special functions, is to generate from (1) a set of one-parameter spectral problems and to explore the spectral inter-relations between this set and the set $\{W_r(\boldsymbol{\lambda})\}_{r=1}^k$. This new set of problems is obtained from (1) by a separation of variables in reverse procedure and, under a certain "definiteness" condition, gives rise to the set of spectral problems

$$\Gamma_r u \equiv \Delta_0^{-1}\Delta_r u = \lambda_r u, \qquad 1 \le r \le k, \tag{5}$$

defined on the tensor product $H^{\otimes} = \bigotimes_{r=1}^k H_r$ and in which we study the existence of simultaneous decomposable common kernels. Such kernels are shown to be eigenvectors for the set $\{W_r(\boldsymbol{\lambda})\}_{r=1}^k$. The operators Δ_r, $0 \le r \le k$ are generated as cofactors of the first row of the determinantal array

$$\begin{vmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_k \\ T_1 & V_{11} & \cdots & V_{1k} \\ \vdots & \vdots & & \vdots \\ T_k & V_{k1} & \cdots & V_{kk} \end{vmatrix}$$

in which $\alpha_r \in \mathbf{C}$, $0 \le r \le k$ are arbitrary.

Throughout the 70s a number of researchers including P. Binding, P. J. Browne, M. Faierman, A. Källström, B. D. Sleeman and others carried forward the programme suggested by Atkinson to the case where the T_r in (1) are unbounded linear operators. Here one shows that a certain system of linear operator equations are uniquely solvable [11] which in turn shows that the operators Γ_r are pairwise commutative. If $E_r(\lambda_r)$ is the resolution of the identity for Γ_r and we form

$$E(\lambda_1 \times \cdots \times \lambda_k) = E_1(\lambda_1) \cdots E_k(\lambda_k),$$

we have the representation

$$f = \int_\sigma E(d\lambda) f$$

where $f \in H^{\otimes}$ and integration is taken over $\sigma = \times_{r=1}^k \sigma(\Gamma_r)$ where $\sigma(\Gamma_r)$ is the spectrum of Γ_r. In the case of pure discrete spectra this amounts to a basic eigenfunction expansion theorem. Various ramifications and extensions of this result were brought together in the monograph [18]. Work on extending multiparameter eigenfunction expansion theorems in a number of directions and under various "definiteness" hypotheses has been a main theme of research in recent years with notable contributions by P. Binding, P. J. Browne, M. Faierman, B. P. Rynne, H. Volkmer and others [17, 19].

The representation theorems developed thus far are given in terms of the spectral measures of the commuting operators Γ_r and in a sense divert attention away from the more accessible spectral measures associated with the original system (1).

In short, the results of Cordes cannot be deduced in their entirety from the work emanating from the Atkinson programme. It is therefore timely that McGhee and Picard have brought attention, in modern form, to the fundamental work of Cordes. There is nothing essentially new in McGhee and Picard's book, but this is unimportant. What is valuable is their careful presentation, in thoughtfully worked sections, of the intricate constructions and detailed arguments necessary for a clear appreciation of Cordes' theory. It raises once more the need to generalise Cordes' results to more than two parameters under the most useful generalisations of the hypotheses (3). There is much here to attract the modern analyst as a rewarding research topic and holding the promise of a multiparameter spectral theory of wide ranging applicability.

Cordes' theory as expounded by McGhee and Picard begins by inducing the system (1) under the hypotheses (3) in the tensor product space (4) to arrive at the induced system.

$$W_r^{\otimes}(\boldsymbol{\lambda}) = 0, \qquad r = 1, 2. \tag{6}$$

Next one forms the "direct sum" space $H = H^{\otimes} \oplus H^{\otimes}$ and reformulates the system (6) as the operator equation

$$T - V \circ \Lambda = 0, \tag{7}$$

defined on H where

$$T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}, \qquad V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$

In addition one constructs the operators Γ_1, Γ_2 as in (5) and considers the operator

$$\Gamma = \Gamma_1 \oplus \Gamma_2.$$

In §§4–6, McGhee and Picard prove that Γ is essentially self-adjoint and that Γ_1, Γ_2 commute, under the basic assumption:

There exist disjoint open intervals $\{I_\nu\}$ and $\{J_\mu\}$, each of which covers the real line except for an at most countable set of points N_I and N_J respectively such that for each pair (ν, μ) for all $(\alpha, \beta) \in I_\nu \times I_\mu$ at least one component of

$$T - V \circ \binom{i}{\alpha} : H \supset D(T) \to H$$

has a bounded inverse on $H^\otimes$ and that at least one component of

$$T - V \circ \binom{\beta}{i} : H \supset D(t) \to H$$

has a bounded inverse on $H^\otimes$.

The main result of §7 and indeed of the whole book may be formulated as follows:

Let $\{\Pi_s(\lambda_s), -\infty < \lambda < \infty\}$ be the spectral family of the self-adjoint operators $\overline{\Gamma}_s : H^\otimes \supset D(\overline{\Gamma}_s) \to H^\otimes$, $s = 1, 2$, and let

$$E(\lambda_1, \lambda_2) \equiv \Pi_1(\lambda_1)\Pi_2(\lambda_2).$$

Define $\Pi(\lambda_1, \lambda_2) \equiv E(\lambda_1, \lambda_2) \oplus E(\lambda_1, \lambda_2) : H \to H$. Suppose the basic assumptions above hold then if $K \Subset (\mathbf{R}\backslash N_I) \times (\mathbf{R}\backslash N_J)$ one has

$$T\Pi(K)f = V \circ \overline{\Gamma}\Pi(K)f$$

for all $f \in H$.

It is then necessary to be precise about what one means by the spectrum of the two-parameter problem. This emerges from §8 as the set

$$\{\Lambda \in \mathbf{C}^2 | \text{ neither compact of } T - V \circ \Lambda \text{ has a bounded inverse}\}.$$

This set is in fact related to the support of the spectral measure $\Pi(\cdot)$, namely the set

$$\{\Lambda \in \mathbf{R}^2 | \Pi(K) \neq 0 \text{ open intervals } K \text{ such that } \Lambda \in K\}.$$

In order to obtain the desired spectral representation theorem it is further assumed that the operator T_1 has a discrete spectrum (if this is not the case the problem remains open). By a local rotation of spectral parameter space one is able (§10) to represent the spectral measure $\Pi(\cdot)$ in terms of computable spectral measures associated with the operators defined in the original spaces H_1 and H_2. Although the resulting spectral representation theorem has a rather complicated form it is widely applicable and for many examples simplifies considerably.

Moving along the mainstream of multiparameter theory it is perhaps unwise to speculate where it will lead. Concerted efforts are being made to develop the algebraic aspects as indicated in the Atkinson programme [12, 16] as well as "inverse" problems (**5**).

It is apparent that there are numerous nonlinear problems arising in physics, engineering and biology involving several parameters and calling for their analysis. Not least is a need for a multiparameter bifurcation theory and methods of solution both analytical and numerical. Tentative results in this direction have been obtained in recent years. There is a vast ocean of nonlinear multiparameter theory awaiting.

References

1. F. M. Arscott, *A treatment of the ellipsoidal wave equation*, Proc. London. Math. Soc. (3) **9** (1959), 21–50.
2. ——, *Two-parameter eigenvalue problems in differential equations*, Proc. London. Math. Soc. (3) **14** (1964), 459–470.
3. F. V. Atkinson, *Multiparameter spectral theory*, Bull. Amer. Math. Soc. **74** (1968), 1–27.
4. ——, *Multiparameter eigenvalue problems: matrices and compact operators*, Academic Press, New York and London, 1972.
5. P. J. Browne and B. D. Sleeman, *Inverse multiparameter eigenvalue problems for matrices*, Proc. Roy. Soc. Edinburgh Sect. (A) **100** (1985), 29–38.
6. R. D. Carmichael, *Boundary value and expansion problems*, Amer. J. Math. **43** (1921), 69–101; ibid. **43** (1921), 232–270; **44** (1922), 129–152.
7. H. O. Cordes, *Über die Spektralzerlegung von hypermaximalen Operatoren die durch Separation der Variablen zer fallen.* I, Math. Ann. **128** (1954), 257–289; II, Math. Ann. (1955), 373–411.
8. A. Erdélyi, *Higher transcendental functions*, Bateman manuscript project vol. 3, McGraw-Hill, New York, Toronto, London, 1955.
9. D. Hilbert, *Grundzuge einer allgemeinsen Theorie der Linearen Integralgleichungen*, Berlin, 1912.
10. H. Isaev, *Lectures on multiparameter spectral theory*, Calgary, 1985.
11. A. Källström and B. D. Sleeman, *Solvability of a linear operator system*, J. Math. Anal. Appl. **55** (1976), 785–793.
12. ——, *Joint spectra for commuting operators*, Proc. Ed. Math. Soc. **28** (1985), 233–248.
13. F. Klein, Math. Ann. **18** (1881), p. 410, but see E. L. Ince, *Ordinary differential equations*, Dover, New York, 1956.
14. F. J. Murray and J. von Neumann, *On rings of operators*, Ann. of Math. (2) **37** (1936), 116–229.
15. A. J. Pell, *Linear equations with two parameters*, Trans. Amer. Math. Soc. **23** (1922), 198–211.
16. B. P. Rynne, *Multiparameter spectral theory and Taylor's joint spectrum in Hilbert space*, Proc. Roy. Soc. Edinburgh Sect. (A) **31** (1988), 127–144.
17. ——, *Uniform convergence of multiparameter eigenfunction expansions*, J. Math. Anal. Appl. (to appear).

18. B. D. Sleeman, *Multiparameter spectral theory in Hilbert space*, Research Notes in Mathematics, vol. 22, Pitman, London, 1978.

19. H. Volkmer, *Multiparameter eigenvalue problems and expansion theorems*, Lecture Notes in Math., vol. 1356, Springer-Verlag, Berlin and New York, 1988.

B. D. SLEEMAN
UNIVERSITY OF DUNDEE

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 22, Number 1, January 1990

0273-0979/90 $1.00 + $.25 per page

Géométrie algébrique réelle, by J. Bochnak, M. Coste and M.-F. Roy. Ergebnisse der Mathematik und ihrer Grenzgebeite, 3 Folge, Band 12, Springer-Verlag, Berlin, Heidelberg, and New York, 1987, x + 375 pp., $118.30. ISBN 3-540-16951-2

Nash manifolds, by Masahiro Shiota. Lecture Notes in Mathematics, vol. 1269, Springer-Verlag, New York, Berlin, Heidelberg, 1987, 273 pp., $20.00. ISBN 0-387-18102-4

One of the basic problems of mathematics is finding the solutions of a system of equations. The easiest case is when the equations are linear and the general form of the set of solutions is well known to every mathematician. The next case one could take is that of polynomials. Suppose we have a collection of polynomials $p_1, p_2, \ldots, p_m$ in n variables with coefficients in some field k. An algebraic set X is the set of common zeroes in k^n of such a set of polynomials, $X = \{x \in k^n \mid p_i(x) = 0, \ i = 1, \ldots, m\}$. Unless one restricts the problem quite a bit (say by taking the p_i's to be quadratic and $n = 3$ and $k =$ the real numbers $\mathbb{R}$) we are nowhere near to completely understanding algebraic sets. Even restricting the polynomials to be quadratic is no help since by adding new variables which are products of the old variables one can reduce the degrees of the polynomials to the point where they are quadratic. (For example $y^2 = x^3$ can be changed to the quadratic $y^2 = xz$, $z = x^2$.)

The study of algebraic sets spawned the field of algebraic geometry which is very active and attracts some of the best mathematicians. However, the natural development of algebraic geometry led to a shift in point of view from the algebraic sets themselves