5. U. Haagerup, *All nuclear C^*-algebras are amenable*, Invent. Math. **74** (1983), 305–319.

6. ——, *A new proof of the equivalence of injectivity and hyperfiniteness for factors on a separable Hilbert space*, J. Functional Analysis **62** (1985), 160–201.

7. B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc., no. 127, Providence, R.I., 1972.

ALLAN M. SINCLAIR
UNIVERSITY OF EDINBURGH

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 22, Number 1, January 1990

0273-0979/90 $1.00 + $.25 per page

Introduction to operator theory and invariant subspaces, by Bernard Beauzamy. North-Holland, Amsterdam, New York, Oxford, Tokyo, 1988, xiv + 358 pp., $84.25. ISBN 0-444-7052-X

The object of study of this monograph is a single continuous linear operator $T: E \to E$, where E is a complex Banach space, and the central question considered is the so-called "invariant subspace problem." We recall that a closed linear subspace $M \subset E$ is invariant for T if $TM \subset M$. The invariant subspace problem asks whether every continuous linear operator T on a Banach space E of dimension ≥ 2 has a nontrivial invariant subspace. (The trivial invariant subspaces are $\{0\}$ and E.) This question was first asked probably by von Neumann in the particular case where E is a Hilbert space, and in this case the problem is still open. When E is a Banach space the answer is negative. Examples of continuous linear operators without invariant subspaces were given first by Enflo [12] on a Banach space built for this purpose. Further examples were given by Beauzamy [6] and Read [16]. Read managed later to produce examples on large families of Banach spaces, including such familiar spaces as l^1 and c (the spaces of summable sequences and convergent sequences, respectively).

One should realize that the invariant subspace problem, basic as it is, was not the only reason for the development of operator theory. In fact, merely knowing that an operator T has nontrivial invariant subspaces does not tell us much about T. Fortunately, when an operator or class of operators is shown to have invariant subspaces, a general structure theory usually emerges, and it is this general structure which is more interesting.

In order to illustrate the previous remarks we consider a few examples. First, let us assume that E is a finite-dimensional Banach space. In this case we can forget for a moment about the Banach space structure, and recall that there is a wonderful structure theorem for linear maps $T: E \to E$; this is the Jordan model theorem. This not only shows that T has invariant subspaces, but in fact it gives a representative (Jordan form) for the similarity equivalence class of T. (Recall that T and T' are similar if $T' = X^{-1}TX$ for some invertible $X: E \to E$.) Still, even the finite-dimensional situation can lead to difficulties when one considers the Banach space structure. Let us say that T, $T': E \to E$ are *unitarily equivalent* if $T' = X^{-1}TX$ for some linear *isometry* X. There is no really good analogue of Jordan's classification theorem for unitary equivalence, even if E is ordinary Euclidean space.

Much of the original interest in operator theory on an infinite-dimensional space E came from the discovery of the compact operators. An operator $T: E \to E$ is said to be compact if it maps any bounded set in E into a set with compact closure. There were good examples of compact operators (many integral operators are compact) and at the same time compact operators behave in many ways like operators on a finite-dimensional space. For instance, if T is compact and λ is a nonzero complex number then the Fredholm alternative holds for $\lambda I - T$ (I denotes the identity operator). That is, $\lambda I - T$ has a continuous inverse if and only if it is one-to-one. Moreover, the dimension of the kernel of $\lambda I - T$ equals the codimension of $(\lambda I - T)E$ in E. In addition, the set $\sigma(T) = \{\lambda : \lambda I - T \text{ is not invertible}\}$, also called the spectrum of T, can only have zero as an accumulation point. Now, $\ker(\lambda I - T)$ is a nontrivial invariant subspace if $\lambda \in \sigma(T)$, $\lambda \neq 0$, so that T does have invariant subspaces if $\sigma(T)$ does not reduce to $\{0\}$. The existence of invariant subspaces when $\sigma(T) = \{0\}$ is harder to prove—this was done by Aronszajn and Smith [3], and they attribute the result to von Neumann if E is a Hilbert space. This invariant subspace theorem led to the development of a triangular structure theory for compact operators (see [13]) and in a parallel way it led to the study of nest algebras (see [4]). The proof of Aronszajn and Smith led to another significant development when E is a Hilbert space. Let $M \subset E$ be a closed subspace and denote by P_M the orthogonal projection of E onto M. Then M is invariant for T if and only if $(I - P_M)TP_M = 0$. An operator T was defined by Halmos [14] to be *quasitriangular* if there exists a sequence $M_1 \subset M_2 \subset \cdots$ of finite-dimensional subspaces of E such that

(i) $\bigcup M_j$ is dense in E; and
(ii) $\lim_{j\to\infty} \|(I - P_{M_j})TP_{M_j}\| = 0$,

where $\|A\|$ denotes the operator norm of A. Aronszajn and Smith used in their proof the quasitriangularity of compact operators, and it was natural to think that it might be possible to prove that all quasitriangular operators have invariant subspaces. Quite amazingly, Apostol, Foiaş, and Voiculescu [2] proved that *nonquasitriangular* operators have nontrivial invariant subspaces. Thus, in Hilbert space, the invariant subspace problem reduces to the study of quasitriangular operators.

The study of compact operators was inspired by the similarity with the finite-dimensional case, while the study of quasitriangularity involved notions that are specifically infinite-dimensional, with no finite-dimensional analogue. One such notion is that of a *Fredholm operator.* A continuous linear operator $T : E \to E$ is called a Fredholm operator if $\ker T$ and E/TE are finite-dimensional spaces. If T is Fredholm but not invertible then either $\ker T$ or TE is a nontrivial invariant subspace. Thus an operator T, for which the set $\sigma_0(T) = \{\lambda \in \sigma(T) : \lambda I - T \text{ is Fredholm}\}$ is not empty, has nontrivial invariant subspaces. The results of Apostol, Foiaş, and Voiculescu imply that T is quasitriangular if $\sigma_0(T)$ is empty. This is again more than an invariant subspace result, and it marked the beginning of a whole new research area.

Suppose now that $T : E \to E$ has an invariant subspace M. Then the restriction $T_1 : M \to M$ of T to M is also a continuous linear operator. Does T_1 have an invariant subspace? If T is compact then T_1 is compact as well, so the Aronszajn-Smith theorem implies a positive answer. Now suppose that E is a Hilbert space and T is a normal operator, i.e., $T^*T = TT^*$, where T^* denotes the Hilbert space adjoint of T. Then T is known to have plenty of invariant subspaces, but the operators T_1 obtained as restrictions of T need not be normal; T_1 is called a *subnormal* operator. The fact that normal operators have invariant subspaces follows from the spectral theorem which shows that in fact a normal operator can be essentially reconstructed from some of its invariant subspaces. There is no spectral theorem for subnormal operators, and it required completely new techniques to show that subnormal operators have nontrivial invariant subspaces. This was done by Brown in [1]. Like in the previous examples, Brown's proof involved a certain structure theory which was subsequently studied and extended. As a result many more classes of operators, on Hilbert as well as Banach spaces, were shown to have nontrivial invariant subspaces. For instance, Brown, Chevreau and Pearcy [9] proved that on a Hilbert space, if $\|T\| \leq 1$ and $\sigma(T) \supset \{\lambda : |\lambda| = 1\}$, then T has nontrivial invariant subspaces.

The book under review is written for the student who wants a rapid introduction to what are—at least in the author's opinion—the most important research subjects in operator theory related to invariant subspaces. The parts that touch on recent subjects are Chapter III (The orbits of a linear operator), Chapter XII (C_1-contractions), Chapter XIII (Positive results), and Chapter XIV (a counter-example to the invariant subspace problem). The idea in Chapter III is to consider invariant closed *subsets* rather than *subspaces* for a linear operator. This leads to the study of the behavior of the orbits $\{x, Tx, T^2x, \ldots\}$ of a linear operator. This idea is also present in Chapter XII, where the author's invariant subspace theorem of [5] is proved. This result applies to operators T such that $\|T\| \leq 1$, the orbit $\{x, Tx, T^2x, \ldots\}$ does not converge to zero for some x, and the backward orbit $\{y, T^{-1}y, T^{-2}y, \ldots\}$ is well behaved for some y. Chapter XIII contains an invariant subspace theorem of Apostol [1] which belongs to the line of work started by Brown and mentioned above. Chapter XIV contains a simple exposition of Read's example of an operator without nontrivial invariant subspaces. This exposition is due to Davie and was previously unavailable in printed form. The remaining parts of the book cover more or less standard topics in functional analysis and operator theory (functional calculus, compact operators, Banach algebras, normal operators, H^p spaces, Sz.-Nagy-Foiaş dilation theory).

The book originates from two courses that the author taught in Paris. An effort has been made to make the book self-contained, but it still has some of the flavor of a set of course notes. The student reading this book will have some basic training in operator theory, but he will not have met a number of fundamental concepts (e.g., that of a Fredholm operator and its index). Apostol's theorem in Chapter XIII is one of the more technically difficult pieces of work in the area started by Brown. There are other results in this subject whose proofs bring more light with less pain for the reader (e.g., the main theorem in [8], which is somewhat less general than Apostol's). I also feel that the book was not sufficiently proofread. There is an abundance of typos and other problems. (For instance, $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ seems to violate the equivalence of (a) and (b) in Exercise 1 of Chapter I.)

In conclusion, I think that this book will be useful to those who want to become familiar with the subjects of Chapters III, XII, XIII, and XIV. The beginning student should however supplement this book with some basic reading, like Dunford and Schwartz [11], Halmos [15], and Douglas [10].

REFERENCES

1. C. Apostol, *Ultraweakly closed operator algebras*, J. Operator Theory **2** (1979), 49–61.
2. C. Apostol, C. Foiaş, and D. Voiculescu, *Some results on nonquasitriangular operators*. IV, Rev. Roumaine Math. Pures Appl. **43** (1973), 487–514.
3. N. Aronszajn and K. Smith, *Invariant subspaces of completely continuous operators*, Ann. of Math. (2) **60** (1954), 345–350.
4. W. Arveson, *Ten lectures on operator algebras*, CBMS Regional Conf. Ser. in Math., no. 55, Amer. Math. Soc., Providence, R. I., 1984.
5. B. Beauzamy, *Sous-espaces invariants de type fonctionnel dans les espaces de Banach*, Acta Math. **144** (1981), 27–64.
6. ——, *Un opérateur sans sous-espace invariant nontrivial: simplification de l'exemple de P. Enflo*, Integral Equations Operator Theory **8** (1985), 314–384.
7. S. Brown, *Some invariant subspaces for subnormal operators*, Integral Equations Operator Theory **1** (1978), 310–333.
8. S. Brown, B. Chevreau, and C. Pearcy, *Contractions with rich spectrum have invariant subspaces*, J. Operator Theory **1** (1979), 123–136.
9. ——, *On the structure of contraction operators*. II, J. Funct. Anal. **76** (1988), 30–55.
10. R. G. Douglas, *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.
11. N. Dunford and J. T. Schwartz, *Linear operators*, Part II: *Spectral theory*, Interscience, New York, 1963.
12. P. Enflo, *On the invariant subspace problem in Banach spaces*, Acta Math. **158** (1987), 213–313.
13. I. C. Gohberg and M. G. Krein, *The theory of Volterra operators in Hilbert space*, "Nauka", Moscow, 1967. (Russian)
14. P. R. Halmos, *Quasitriangular operators*, Acta Sci. Math. (Szeged) **29** (1968), 283–293.
15. ——, *A Hilbert space problem book*, Second ed., Springer-Verlag, New York, 1982.
16. C. Read, *A solution to the invariant subspace problem*, Bull. London Math. Soc. **16** (1984), 337–401.

HARI BERCOVICI
INDIANA UNIVERSITY