and others (see Chapter XVII of [1]) focused on the problem of determining what abelian groups could be realized as the additive group of an Artinian ring.

The book under review addresses the group $U(R)$ in several contexts where this group has traditionally been of interest. The first context in which $U(R)$ was extensively studied was that in which R is the ring of algebraic integers of a finite algebraic number field K. The principal result in this area is the Dirichlet-Chevalley-Hasse Unit Theorem, which states that $U(R)$ is the direct product of a specified group and a free abelian group of specified finite rank; in particular, $U(R)$ is finitely generated. An important special case that lies within this setting is that in which $K = Q(\varepsilon_n)$ is the cyclotomic field of nth roots of unity over the rational field Q, and hence $R = Z[\varepsilon_n]$; the author treats this case in Chapter 3. Other topics covered are the unit groups of fields, division rings and group rings; moreover, Chapter 7 is devoted to consideration of finite generation of $U(R)$.

Karpilovsky's book brings together a broad range of topics from group theory, commutative and noncommutative ring theory, field theory, and algebraic number theory. The quality of exposition in the text is quite good; the author has done a praiseworthy job in making the material accessible to a knowledgable, but nonspecialist, reader. In the process, some generality and depth of coverage has been sacrificed in order to broaden the audience for the book. Overall, the book can be highly recommended to a reader interested in learning about a wide range of topics in which unit groups have historically played a significant role.

Reference

1. L. Fuchs, *Infinite abelian groups*, vol. II, Academic Press, New York, 1973.

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Theory of reproducing kernels and its applications, By Saburou Saitoh, Longman Scientific and Technical, 157 pp., 1988, $57.95. ISBN 0-582-03564-3

Let S be a set and $\mathscr{H}$ a Hilbert space; a mapping κ from S to $\mathscr{H} : x \to k_x$ gives rise to a kernel function

$$K(x, y) = (k_y, k_x)$$

defined on $S \times S$. It is clear that considerable information about the map κ is given by the kernel function; it prescribes the length of each vector k_x and the inner product between any two such vectors. It follows that the image of S under κ is determined by $K(x, y)$ up to a (linear) isometry.

We have also another mapping: κ^* carries every element f of $\mathscr{H}$ into a function defined on S by the formula

$$f(x) = (f, k_x)$$

and the collection of such functions forms a linear space. If we suppose as is often the case, that the null space of κ^* is trivial the mapping is one-to-one and we can give the function $f(x)$ the (quadratic) norm $\|f\|$. This space of functions, denoted by our author $\mathscr{H}_K$ is then a Hilbert space of functions defined on S, and if $K_z(x) = (k_z, k_x) = K(x, z)$ then $f(z) = (f, K_z)$, the inner product being taken in the function space $\mathscr{H}_K$. It follows that the valuation of f in $\mathscr{H}_K$ at a point z in S is a continuous linear function on that space.

Indeed, if we are given any Hilbert space consisting of functions defined on a set S where the point evaluations are continuous linear functionals, a well-known theorem of F. Riesz guarantees that such functionals are inner products with a unique element K_z in the space. We have then a mapping κ of S into the Hilbert space in question just as above.

Let us note that the function $K(x, y) = (k_y, k_x)$ is a sort of Gram's matrix. Many years ago E. H. Moore considered functions $K(x, y)$ defined on $S \times S$ and introduced the definition of a *positive matrix*: such a function is a positive matrix if, for every finite set of points x_i in S and equally many complex numbers c_i,

$$\text{(PM)} \qquad \sum_i \sum_j K(x_i, x_j)\bar{c}_i c_j \geq 0.$$

In the case we are considering this double sum is merely the square of the norm of the element $\sum_i c_i k_{x_i}$. Thus our kernel function is a positive matrix. Finally Moore showed that every positive matrix $K(x, y)$ is in fact the kernel function of a map κ. For this purpose we consider the set of all finite formal sums $f = \sum c_i k_{x_i}$ with complex coefficients endowed with the quadratic semi-norm $\|f\|^2$ defined by (PM) above. Taking this space modulo the null set of the semi-norm gives us a pre-Hilbert space whose completion is $\mathscr{H}^*$. The mapping K from S to $\mathscr{H}^*$ defined so that the image of

x in S is the equivalence class containing k_x then has $K(x, y)$ for its kernel function.

Thus the circle of ideas is closed and we have a complete equivalence between kernel functions, mappings κ, functional Hilbert spaces with continuous evaluations and positive matrices.

The study of kernel functions was essentially initiated by S. Bergman who noticed that if $\{g_n\}$ was a complete orthonormal set in a separable $\mathscr{H}$, then $k_x = \sum_n (k_x, g_n) g_n$ where the Fourier coefficients $(k_x, g_n) = \bar{g}_n(x)$ are square-summable. Thus

$$K(x, y) = (k_y, k_x) = \sum_n g_n(x)\bar{g}_n(y)$$

where the series converges absolutely. This result holds independently of the choice of the complete orthonormal set.

The term "reproducing kernel" is used for the kernel function of the "reproducing kernel space" $\mathscr{H}_K$. Such spaces occur in many areas of analysis. For example, on a locally compact abelian group, the functions of positive type are those continuous functions $\phi(x)$ such that $K(x, y) = \phi(x - y)$ is a positive matrix. Similarly, the covariance functions in the study of stochastic processes are reproducing kernels. But the core of the subject is in its application to the study of analytic functions of one or more variables initiated by Bergman.

In the book under review the author considers almost exclusively spaces of analytic functions. He covers the classical work of Bergman and Schiffer on reproducing kernel spaces as well as a great deal of more recent work. The extent of his coverage in a small volume often prevents him from giving full proofs, and he contents himself with describing the situation and giving complete references to the existing work. Naturally the author is most interested in fields to which he has made extensive contributions himself.

One of these is the study of integral operators arising when the mapping κ^* of S has its range in an L^2-space; κ^* is then an integral operator from that L^2-space to $\mathscr{H}_K$. Under the hypothesis that the norm in $\mathscr{H}_K$ is given by an L^2-norm relative to some measure μ on S the author determines a formula for the inverse transform. Another fascinating topic, close to this reviewer's heart, is the study of the Pick-Nevanlinna interpolation.

It should be emphasized that Professor Saitoh's book is not for the beginner. It will be appreciated by the expert to whom it is addressed, not least for its remarkably complete bibliography of the whole subject of reproducing kernels.

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Geometric inequalities, by Yu. D. Burago and V. A. Zalgaller. (Translated by A. B. Sossinsky), Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1988, xiv + 331 pp., $97.00. ISBN 3-540-13615-0

This volume presents us with a masterful treatment of a subject that is not so easily treated. The basic difficulty is that "geometric inequalities" is not so much a subject as a collection of topics drawing from diverse fields and using a wide variety of methods. One can therefore not expect the kind of cohesiveness or of structural development that is possible in a single-topic book. At most one hopes for a broadly representative selection of theorems organized by approach or content, with a good accounting of each and ample references for following up in any given direction; and that is just what we get.

All the classical topics are found here: the isoperimetric inequality in its many guises, the Brunn-Minkowski inequality with its various consequences, area and volume bounds of different kinds. There are also many inequalities involving curvatures: Gauss, mean, Ricci, etc. The methods include those of differential geometry, geometric measure theory, and convex sets. In each of these areas, the book is right up to date, including the latest results to the time of writing.

In addition to these classical topics, there are some more modern ones. Chapter 3 includes an extended and illuminating discussion of various notions of area and measure, including the newer approaches dating from the 1960s: the *perimeter* of Caccioppoli and de Giorgi, *integral currents* of Federer and Fleming, Almgren's *varifolds*. Their relative merits and disadvantages are carefully and even-handedly pointed out. Chapter 6, on Riemannian manifolds, provides a complete proof of Margulis' Theorem giving a lower bound for the volume of a compact negatively curved manifold in terms of a lower bound on the curvature.