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PRODUCT FORMULAS, HYPERGROUPS, AND THE JACOBI POLYNOMIALS

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If $\mathscr{P} = \{p_n\}_{n\in\mathbf{N}_0}$ ($\mathbf{N}_0 = \{0, 1, 2, \dots\}$) is a sequence of orthogonal functions on a real interval I, we say that $\mathscr{P}$ has a *product formula* if for each s, t in I, there is a Borel measure $\mu_{s,t}$ with $\operatorname{supp}(\mu_{s,t}) \subseteq I$ such that

$$\int_I p_n \, d\mu_{s,t} = p_n(s)p_n(t) \tag{1}$$

for every n in $\mathbf{N}_0$. Such formulas are important because they give rise to a variety of measure algebras and the means to study their harmonic analysis. An important class of such formulas was established by Gasper [8] for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ which are orthogonal on $[-1, 1]$ with respect to the weight $(1-x)^\alpha \times (1+x)^\beta \, dx$. These include Chebyshev, Legendre, and ultraspherical or Gegenbauer polynomials as special cases. The product formula for Jacobi polynomials has 1 as an *identity* in the sense that for all t in $[-1, 1]$ $\mu_{1,t}$ is the unit point mass concentrated at t, and it has *continuous support* in the sense that $\operatorname{supp}(\mu_{s,t})$ is a continuous function of (s, t). Moreover, the measures $\mu_{s,t}$ are all positive if and only if

$$\alpha \ge \beta > -1 \text{ and either } \beta \ge -1/2 \text{ or } \alpha + \beta \ge 0. \tag{2}$$

It is natural to ask which orthogonal polynomials have such product formulas. The answer is a converse to Gasper's result:

Theorem 1. *If a family $\mathscr{P}$ of orthogonal polynomials has a product formula with identity, continuous support, and nonnegative measures $\mu_{s,t}$ then up to a linear change of variables, the members of $\mathscr{P}$ are Jacobi polynomials with parameters α and β satisfying equation* (2).

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The proof of Theorem 1, as well as some of its applications will require the notion of hypergroups. A hypergroup is a measure algebra more general then the convolution measure algebra associated with a group (for instance, a convolution of point masses need not be a point mass), but with enough structure to make harmonic analysis possible.

To be precise, let H be a locally compact Hausdorff space and let $M(H)$ denote the bounded Borel measures on H; if $\mu \in M(H)$, $\operatorname{supp}(\mu)$ is the support of μ. The unit point mass concentrated at s is indicated by δ_s; $C(H)$ is the space of continuous complex-valued functions on H; and $C_c(H)$ consists of all f in $C(H)$ with compact support.

If $M(H)$ is a Banach algebra with multiplication $*$ (called a convolution), then $(H, *)$ is a *hypergroup* if the following axioms are satisfied:

(1) A convolution of probability measures is a probability measure.

(2) The mapping $(\mu, \nu) \to \mu * \nu$ is continuous from $M(H) \times M(H)$ into $M(H)$ where $M(H)$ is given the weak topology with respect to $C_c(H)$.

(3) There is an element $e \in H$ such that $\delta_e * \mu = \mu * \delta_e = \mu$ for every $\mu \in M(H)$.

(4) There is a homeomorphic mapping $s \to s^\vee$ of H into itself such that $s^{\vee\vee} = s$ and $e \in \operatorname{supp}(\delta_s * \delta_t)$ if and only if $t = s^\vee$.

(5) For $\mu, \nu \in M(H)$ $(\mu * \nu)^\vee = \nu^\vee * \mu^\vee$ where $\mu^\vee$ is defined by

$$\int_H f(s)\, d\mu^\vee(s) = \int_H f(s^\vee)\, d\mu(s).$$

(6) The mapping $(s, t) \to \operatorname{supp}(\delta_s * \delta_t)$ is continuous from $H \times H$ into the space of compact subsets of H as topologized in [14].

See [5, 6, 9 and 16] for more about hypergroups.

There is a natural connection between product formulas and hypergroups. Suppose the hypotheses of Theorem 1 are satisfied. Define a product $*$ on $M(I)$ by

$$\int_I f d(\nu * \lambda) = \int_I \int_I \left(\int_I f\, d\mu_{s,t} \right) d\nu(s)\, d\lambda(t), \tag{3}$$

then $(I, *)$ is a hypergroup if we require the additional conditions: (i) there is $e \in I$ such that $p_n(e) = 1$ $(n \in \mathbf{N}_0)$, and (ii) $0 \in \operatorname{supp}(\mu_{s,t})$ if and only if $s = t$ [16]. $\mathscr{P}$ is the set of characters for this hypergroup in the sense that (1) and (3) imply

$$\int_I p_n\, d(\nu * \lambda) = \left(\int_I p_n\, d\nu \right) \left(\int_I p_n\, d\lambda \right) (n \in \mathbf{N}_0).$$

The hypergroups arising from the ultraspherical polynomials are useful in studying certain stochastic processes on the sphere (see also [9]).

Theorem 1 immediately yields the following characterizations of two classes of hypergroups (see [5] for definitions):

Theorem 2. *If* $(H, *)$ *is a hypergroup with* H *a real interval which has polynomial characters of every degree, then up to a linear change of variables,* $(H, *)$ *is one of the Jacobi polynomial hypergroups* $(H, *; \alpha, \beta)$ *with parameters* α *and* β *satisfying equation* (2).

Theorem 3. *The only strong polynomial hypergroups* (*see* [9]) *are those that arise from the Jacobi polynomials with parameters* α *and* β *satisfying* (2).

Theorem 3 gives further credence to Heyer's remark [9, 3.7] that strong hypergroups are fairly rare. Thus results for strong polynomial hypergroups [11, Theorem 4], or for hypergroups with polynomial characters are no more than the corresponding results for the hypergroups arising from Jacobi polynomials for which there is already an extensive literature; e.g., [3, 4], and the references cited there. It is possible to give explicit formulas for the parameters α and β and the p_n. We first observe that one consequence of Theorem 1 is that e must be one of the endpoints of I [16]:

Theorem 4. *Let* $\mathscr{P}$ *satisfy the hypotheses of Theorem* 1 *with identity* e*; let* a *be the other endpoint of* I*, and let* $d_n = \frac{1}{2}p_n'(e)/(e-a)$*, then* $\alpha = (d_2 - 2d_1)^{-1} - 1$*,* $\beta = (2d_1 - 1)(d_2 - 2d_1)^{-1} - 1$*, and* $p_n(t) = R_n^{(\alpha,\beta)}(t) = \binom{n+\alpha}{n}^{-1} P_n^{(\alpha,\beta)}\left(\frac{a+e-2t}{a-e}\right)$.

We note that there are systems of orthogonal polynomials besides the Jacobi polynomials which have product formulas.

1. The generalized Chebyshev polynomials have a product formula [10] which does not have support continuity since $\operatorname{supp}(\mu_{1,-1}) = \{-1\}$ but $\operatorname{supp}(\mu_{t,-t}) = [-1, 1-2t^2] \cup [2t^2-1, 1]$ (see [12, p. 207]; (in that article, the notation for $\mu_{s,t}$ is $p_s * p_t$).

2. The continuous q-Jacobi polynomials [15] on $H = [-1, 1]$ have a product formula with a nonnegative absolutely continuous measure for all $s, t \in H$ and the support of the measure is always all of H, hence there can be no identity.

Outline of proof of Theorem 1. The proof has two parts. First a technique inspired by [13] and exploited in [5] shows that the members of $\mathscr{P}$ are the eigenfunctions of a second order linear differential operator. That is $y = p_n(t)$ satisfies

$$qy'' + py' = \lambda_n y \tag{4}$$

with $\lambda_n = p_n'(e)$, $p(t) = p_1(t)$, $q(t) = [\lambda_2/p_2''][p_2(t) - p_1(t)] - [p_1(t)/\lambda_1][p_1(t) - 1]$. Secondly we employ a result of Bochner [2] to show that the differential equation must in fact be the one associated with the Jacobi polynomials.

If $q \equiv 0$, the solutions of equation (4) are $p_n(x) = x^n$ which are not orthogonal on any interval since these polynomials do not have simple zeros (see [18, Theorem 3.3.1]).

If q is a nonzero constant, $p_n(x) = H_n(x)/H_n(e)$, where H_n is the nth degree Hermite polynomial. The condition $q'' = 0$ entails

$$p_2'(e) = 2p_1'(e) \tag{5}$$

which leads to a contradiction when one attempts to solve equation (5) for e.

If q has degree exactly one then p_n is the normalized Laguerre polynomial $L_n^\alpha(x)/L_n^\alpha(e)$. Once more, equation (5) must hold, but this time it can be solved to obtain $e = 0$. The nonexistence of a product formula in this case follows from [1], (Theorem 6 and the remarks following).

Thus q must have degree exactly two. A linear change of variables transforms equation (4) into one of the forms

$$x^2y'' + (\delta + \varepsilon x)y' + \lambda y = 0. \tag{6}$$

$$x(1-x)y'' + (\delta + \varepsilon x)y' + \lambda y = 0. \tag{7}$$

We shall eliminate the possibility of equation (6) by showing that if the differential operator $L = t^2(d^2/dt^2) + (\delta + \varepsilon t)d/dt$ has polynomial eigenvectors, they cannot be orthogonal. Bochner [2] considers two cases: $\delta = 0$ and $\delta \neq 0$.

If $\delta = 0$, then L has polynomial eigenfunctions provided $\varepsilon = 1 - k$, $k = 1, 2, \ldots$, in which case the eigenfunctions are of the form $p_n(x) = a_nx^n + b_nx^{k-n}$. These cannot be orthogonal polynomials since for $n > k$, the zeros of p_n are not distinct (cf. [18, Theorem 3.3.1]).

If $\delta \neq 0$, it is no loss of generality to consider only $\delta + \varepsilon t = (k+1)t - 1$. Then L has polynomial eigenfunctions unless k is a negative integer, and if $-k \notin \mathbf{N}_0$ is fixed, these are given by

$$P_n(t) = \sum_{\nu=0}^{n} \nu! \binom{n}{\nu} \binom{-n-k}{\nu} t^\nu.$$

It can be shown that these polynomials are not orthogonal because they do not satisfy an appropriate three-term recurrence (cf. [17, Theorem 1]).

Having eliminated all other possibilities, we conclude that the differential equation (4) must have been transformed into equation (7). If in that equation we make the change of variables $x = 2t-1$, we obtain a differential equation satisfied by $y = P_n^{(\alpha,\beta)}(x)$ (see [18, equation (2.1)]) with $\alpha = -2\varepsilon - 2\delta - 1$, $\beta = 2\delta - 1$, and $\lambda = n(n+\alpha+\beta+1)/4$, and the proof is complete.

Remark. Slight modifications can be made in the proof to allow the hypotheses of Theorem 1 to be weakened as follows:

1. Instead of assuming that e is an identity, it is enough to ask that for each $t \in I$, $\mu_{e,t}$ be concentrated on a single point. (It is not necessary to require that $\mu_{e,t}$ be a unit mass or concentrated on the point t.)
2. The support continuity may be replaced by

$$\lim_{s\to e}(\operatorname{diam}(\operatorname{supp}\mu_{s,t})) = 0.$$

3. The combination of support continuity and nonnegativity may be replaced by the single condition $\int_I (r-t)^n \, d\mu_{s,t}(r) = o(s-e)$ for $n > 2$.
4. The condition that the polynomials satisfy a product formula can be replaced by the assumption that the weighted polynomials $m(t)p_n(t)$ satisfy a product formula, where $m(t)$ is a fixed positive function on I.

Proof of Theorems 2 *and* 3. The hypotheses of these theorems are by definition stronger than the hypotheses of Theorem 1, since the characters of a commutative hypergroup are necessarily orthogonal [6, Theorem 3.5]. The range for the parameters is the intersection of those given by Gasper in his studies [7, and **8**] of the two Jacobi convolution structures.

Proof of Theorem 4. The linear transformation $x = \phi(t) = (a+e-2t)/(a-e)$ maps I onto $[-1, 1]$ and carries e to 1. Thus $p_n(t) = R_n^{(\alpha,\beta)}(\phi(t))$, and the relations are obtained by referring to the explicit formulas for the first two moments associated with Jacobi polynomials as given in [5, equations (1.10) and (1.11)].

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