SELF-MAPS OF CLASSIFYING SPACES
OF COMPACT SIMPLE LIE GROUPS

STEFAN JACKOWSKI, JAMES E. MCCLURE, AND BOB OLIVER

ABSTRACT. We describe here the set $[BG, BG]$ of homotopy
classes of self-maps of the classifying space BG, for any com-
pact connected simple Lie group G. In particular, we show
that two maps $f, f' : BG \to BG$ are homotopic if and only
if they are homotopic after restricting to the maximal torus of
G; or equivalently if and only if they induce the same homo-
morphism in rational cohomology. In addition, we identify the
homotopy types, up to profinite completion, of the components
of the mapping space $\mathrm{map}(BG, BG)$.

The most central concern of homotopy theory is the classifi-
cation, up to homotopy, of maps between topological spaces. It
has long been suspected that maps between classifying spaces pro-
vide a particularly favorable special case of this problem, in which
explicit results can be expected. In this paper we announce a com-
plete classification of the self-maps of the classifying space BG,
when G is any compact connected simple Lie group.

When G and Γ are arbitrary compact Lie groups, then
$[BG, B\Gamma]$ will denote the set of unbased homotopy classes of
maps from BG to $B\Gamma$. It is natural to ask how closely this
set is related to the set $\mathrm{Hom}(G, \Gamma)$ of homomorphisms from
G to Γ. For any inner automorphism $\alpha \in \mathrm{Inn}(\Gamma)$, $B\alpha$ is ho-
motopic to the identity on $B\Gamma$. It is thus convenient to write
$\mathrm{Rep}(G, \Gamma) = \mathrm{Hom}(G, \Gamma)/\mathrm{Inn}(\Gamma)$; and ask when the map

$$B : \mathrm{Rep}(G, \Gamma) \to [BG, B\Gamma]. \qquad B(\rho) = B\rho$$

is a bijection.

When G and Γ are both finite (or even discrete), then B is
easily seen to be bijective. A much deeper result, due to Dwyer and
Zabrodsky [2], says that B is bijective whenever G is a p-group,
(and Γ is compact Lie). This was extended by Notbohm [8] to the
case where G is a p-toral group: i.e., where G has toral identity

component and $\pi_0(G)$ is a p-group. These theorems depend on the generalized Sullivan conjecture, proven by Miller, Carlsson, and Lannes using different approaches [16, 17, 18].

Our goal here is to apply the theorems of Dwyer-Zabrodsky and Notbohm to study the sets $[BG, BG]$ when G is a connected, compact, simple Lie group. The first important contribution to this was given by Sullivan [11]. For $G = SU(n)$ and k prime to $n!$, he showed that the kth power map on BT ($T \subseteq G$ the maximal torus) can be extended to an "unstable Adams operation" $\psi^k : BSU(n) \to BSU(n)$. Since the kth power map cannot be extended to a homomorphism, $B : \mathrm{Rep}(G, G) \to [BG, BG]$ is far from surjective in these cases—but the sets $[BG, BG]$ still have very simple descriptions.

Theorem 1. *Let G be a compact connected simple Lie group with maximal torus T and Weyl group W. Then, for any pair of maps $f, f' : BG \to BG$, the following are equivalent:*

   (1) *f and f' are homotopic;*
   (2) *$f \circ B\iota \simeq f' \circ B\iota : BT \to BG$ ($\iota : T \hookrightarrow G$);*
   (3) *$H^*(f; \mathbf{Q}) = H^*(f'; \mathbf{Q})$.*

Furthermore, the image of the injective restriction map

$$[BG, BG] \rightarrowtail [BT, BG]$$

consists of all maps $B\rho$ for ρ a composite

$$T \xrightarrow{\ t \mapsto t^k\ } G \xrightarrow{\alpha} G,$$

where $(k, |W|) = 1$ and $\alpha \in \mathrm{End}(G)$.

What is new in this theorem is the equivalence of points (1) and (2): note the analogy with the situation for group representations. The equivalence of (2) and (3) follows easily from [8].

The image in $[BT, BG] \cong \mathrm{Rep}(T, G)$ of the restriction map was already known. As mentioned earlier, the existence of maps $\psi^k : BG \to BG$ (for k prime to $|W|$) which extend the kth power map on BT was shown by Sullivan when $G = SU(n)$—and his result was extended to arbitrary simple G by Wilkerson [12]. Conversely, Hubbuck [3, 4] showed that for every $f : BG \to BG$, $H^*(f, \mathbf{Q}) = H^*(B\alpha \circ \psi^k; \mathbf{Q})$ for some α and k; and Ishiguro [5] showed (using [2]) that either $k = 0$ or $(k, |W|) = 1$. Hubbuck's result also follows as a consequence of Notbohm's theorem.

Various examples constructed by the authors show that Theorem 1 does not generalize to the case of $[BG, B\Gamma]$ for arbitrary

(or even connected) G and Γ. For example, there are homomorphisms $\rho, \rho' : SU(3) \to SO(8)$ such that $B\rho$ and $B\rho'$ induce the same homomorphism in rational cohomology, but are not homotopic. Also, the Borel construction can be applied to the tangent bundle of any G-action on a contractible oriented n-manifold to give a vector bundle over BG—and hence a map from BG to $BSO(n)$. Thus, the many examples of exotic G-actions on $\mathbf{R}^n$ can be used to construct exotic elements in $[BG, BSO(n)]$.

Now let $G \supseteq T$ be as in Theorem 1. A map $f : BG \to BG$ will be said to have "type k" if the following diagram commutes in rational cohomology (or equivalently [8] up to homotopy):

$$
\begin{array}{ccc}
BT & \xrightarrow{B\psi k} & BT \\
{\scriptstyle B\iota}\downarrow & & \downarrow{\scriptstyle B\iota} \\
BG & \xrightarrow{\quad f \quad} & BG
\end{array}
\qquad
\begin{array}{l}
(\iota : T \hookrightarrow G) \\[2mm]
(\psi k(t) = t^k)
\end{array}
$$

More generally, for any outer automorphism $\alpha \in \mathrm{Out}(G)$, f will be said to have type (α, k) if $B\alpha^{-1} \circ f$ has type k.

Note that for simple G, $\mathrm{End}(G)/\mathrm{Inn}(G) \cong \{0\} \coprod \mathrm{Out}(G)$—since any endomorphism of G is either trivial or an automorphism. Theorem 1 can now be reformulated as follows:

Theorem $1'$. *For any compact connected simple Lie group G with Weyl group W, the correspondence $f \leftrightarrow \mathrm{type}(f)$ defines a bijection*

$$
[BG, BG] \xleftrightarrow{\;\cong\;} \{0\} \coprod [\mathrm{Out}(G) \times \{k > 0 : (k, |W|) = 1\}].
$$

Consider the important special case $G = SU(n)$. Note that $\mathrm{Out}(SU(2)) = 1$, and that $\mathrm{Out}(SU(n)) = \{1, \kappa\}$ for $n \geq 3$ (where κ denotes complex conjugation). Since $B\kappa$ is an unstable Adams operation of type -1, Theorem $1'$ implies that every self-map of $BSU(n)$ is of type k for some k—and that any two maps of type k are homotopic.

The first case of Theorem $1'$—when $G = SU(2)$—was proven by Mislin [7]. Later, Dwyer and Mislin [1] gave a complete description of the homotopy type of the space of pointed self-maps of $BSU(2)$ (see also Theorem 2 below).

In the proof of Theorem 1 outlined below, we use not only the theorems of Dwyer, Zabrodsky, and Notbohm about the sets $[BP, B\Gamma]$ for p-toral P, but also their computation of the p-adic homotopy type of the individual connected components of $\mathrm{map}(BP, B\Gamma)$. In order to do this, Dwyer and Zabrodsky introduced the following construction. For any G and Γ, and any $\rho : G \to \Gamma$, let $C_\Gamma(\rho)$ denote the centralizer of $\mathrm{Im}(\rho)$ in Γ. Then

multiplication in Γ defines a homomorphism $(\mathrm{incl}, \rho) : C_\Gamma(\rho) \times G \to \Gamma$, which induces a map
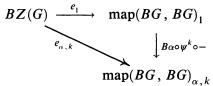
$$e_\rho : BC_\Gamma(\rho) \to \mathrm{map}(BG, B\Gamma)_{B\rho}.$$

Here, $\mathrm{map}(BG, B\Gamma)_{B\rho}$ denotes the component containing $B\rho$.

As usual, the case where G and Γ are both finite is easy: e_ρ is then a weak homotopy equivalence for each ρ. What is relevant for our results is that when G is p-toral, then e_ρ induces an isomorphism of mod p cohomology for each ρ (see [2 and 8]).

Now assume again that G is connected and simple. When ρ is the trivial or identity endomorphism of G, then e_ρ takes the form

$$e_0 : BG \to \mathrm{map}(BG, BG)_0 \quad \text{and} \quad e_1 : BZ(G) \to \mathrm{map}(BG, BG)_1,$$

where $Z(G)$ is the center of G. For every k, α, $\mathrm{map}(BG, BG)_{\alpha, k}$ will denote the space of maps of type (α, k). For arbitrary $\alpha \in \mathrm{Out}(G)$ and $(k, |W|) = 1$, we let $e_{\alpha, k}$ denote the following composite:

$$
\begin{array}{ccc}
BZ(G) & \xrightarrow{\;\;e_1\;\;} & \mathrm{map}(BG, BG)_1 \\
& \underset{e_{\alpha, k}}{\searrow} & \Big\downarrow{B\alpha \circ \psi^k \circ -} \\
& & \mathrm{map}(BG, BG)_{\alpha, k}
\end{array}
$$

for some map ψ^k of type k. Our description of the profinite homotopy type of $\mathrm{map}(BG, BG)$ is completed by the following theorem.

Theorem 2. *For any connected simple G, the maps e_0, and $e_{\alpha, k}$ for each α and k, induce isomorphisms of homology with any finite coefficients.*

We now outline the proof of Theorems 1 and 2. With the help of Sullivan's arithmetic square [11] for BG, the study of $\mathrm{map}(BG, BG)$ can be reduced to that of the mapping spaces $\mathrm{map}(BG, BG\hat{p})$ for all primes p: where $(-)\hat{p}$ denotes the p-adic completion.

The key step is to approximate BG at each prime p, as a homotopy colimit of BP's for certain p-toral subgroups $P \subseteq G$. This can in fact be done for any compact Lie group G, extending results in [6]. The indexing category for these limits is the full subcategory $\mathscr{R}_p(G)$ of the category of G-spaces, whose objects are the orbits G/P for which

(i) $P \subseteq G$ is p-toral and

(ii) $N(P)/P$ is finite and contains no nontrivial normal p-subgroups.

This is a category with finitely many isomorphism classes of objects and finite morphism sets. Let $\mathscr{I} : \mathscr{R}_p(G) \to G\text{-spaces}$ be the inclusion functor, and let $EG \times_G \mathscr{I}$ be its composite with the Borel construction. Note that $(EG \times_G \mathscr{I})(G/H) = EG \times_G G/H$ is homotopy equivalent to BH.

Theorem 3. *For any compact Lie group G and any prime p,* $\mathrm{hocolim}_{\mathscr{R}_p(G)}(\mathscr{I})$ *is* $\mathbf{Z}/p$-acyclic, *and hence*

$$\mathrm{hocolim}_{\mathscr{R}_p(G)}(EG \times_G \mathscr{I}) \simeq_p EG \times_G * \simeq BG.$$

The main step when proving Theorem 3 is to construct a finite dimensional $\mathbf{Z}/p$-acyclic G-complex X, all of whose orbit types are in $\mathscr{R}_p(G)$. This is done using the approach in [9] and [10]. Since the fixed point set $(\mathrm{hocolim}_{\mathscr{R}_p(G)}(\mathscr{I}))^P$ is contractible for any G/P in $\mathscr{R}_p(G)$, there is a G-equivariant $\mathbf{Z}/p$-homology equivalence $X \to \mathrm{hocolim}_{\mathscr{R}_p(G)}(\mathscr{I})$. Thus, $\mathrm{hocolim}_{\mathscr{R}_p(G)}(\mathscr{I})$ is $\mathbf{Z}/p$-acyclic since X is.

Recall that the main problem is to show that (2) implies (1) in Theorem 1; i.e., that any two maps $f, f' : BG \to BG$ which are homotopic on BT are homotopic on BG. For that, it suffices to show that they are homotopic inside $BG\hat{p}$ for each prime p. Using the theorems of Hubbuck [3, 4] and Notbohm [7], we first show that if f and f' are homotopic on BT, then they are homotopic on BP for each prime p and each p-toral subgroup $P \subseteq G$.

At this point, higher derived functors of inverse limits over $\mathscr{R}_p(G)$ begin to play a role. By a result of Wojtkowiak [14], if

$$\varprojlim_{\mathscr{R}_p(G)}{}^i (\pi_j(\mathrm{map}(EG \times_G \mathscr{I}, BG\hat{p})_f) = 0$$

for all $i, j > 0$, then the inverse limit

$$\varprojlim_{\mathscr{R}_p(G)} (\mathrm{map}(EG \times_G \mathscr{I}, BG\hat{p})_f)$$

is also connected, and hence f' is homotopic to f. Also, in this case,

$$\pi_*(\mathrm{map}(BG, BG\hat{p})_f) \cong \varprojlim_{G/P \in \mathscr{R}_p(G)} (\pi_*(\mathrm{map}(EG \times_G \mathscr{I}, BG\hat{p})_f)).$$

Note that

$$\varprojlim_{\mathscr{R}_p(G)}^{i}(\pi_j(\mathrm{map}(EG \times_G \mathscr{I}, BG\hat{p})_f))$$

$$= \varprojlim_{G/P \in \mathscr{R}_p(G)}^{i}(\pi_j(\mathrm{map}(BP, BG\hat{p})_{f|BP}))$$

and similarly for the other limits.

The remaining step is now to show that these higher limits all vanish. The inverse systems are made explicit by the isomorphism

$$\pi_*(\mathrm{map}(BP, BG\hat{p})_f) \cong \pi_*(BC_G(\rho)\hat{p})$$

of [2] and [8]: for $\rho: P \to G$ such that $B\rho \simeq f \mid BP$. In particular, if (α, k) is the type of f (and f'), we get that

$$\pi_*(\mathrm{map}(BP, BG\hat{p})_{f|BP}) \simeq \begin{cases} \pi_*(BC_G(P))\hat{p} & \text{if } k > 0, \\ \pi_*(BG)\hat{p} & \text{if } k = 0. \end{cases}$$

For components of type (α, k) with $k \neq 0$, Theorem 1 and 2 now follow from the following lemma:

Lemma 4. *If G is simple and p is any prime, then*

$$\varprojlim_{G/P \in \mathscr{R}_p(G)}^{i}(\pi_j(BC_G(P))\hat{p}) \cong \begin{cases} \pi_j(BZ(G))\hat{p} & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

The proof of Lemma 4 uses a reinterpretation of derived functors over $\mathscr{R}_p(G)$. For any (contravariant) functor $F: \mathscr{R}_p(G) \to$ *Groups*, $\varprojlim_{\mathscr{R}_p(G)}^{*}(F) \cong H_G^*(\mathrm{hocolim}_{\mathscr{R}_p(G)}(\mathscr{I}); F)$. Here, $H_G^*(-; F)$ denotes equivariant cellular cohomology (see [13]). This last group is closely related to $H_G^*(X; F)$, where X is the finite dimensional acyclic G complex described after Theorem 3. The lemma is then proven by computing these cohomology groups for the given functors on $\mathscr{R}_p(G)$, using among other things the proof of the Conner conjecture in [10].

For f of type 0, Theorems 1 and 2 are special cases of the following theorem.

Theorem 5. *Let G and Γ be any two compact Lie groups, and let T be a maximal torus in G.*

(i) *A map $f: BG \to B\Gamma$ is null homotopic if and only if $f \mid BN(T)$ is null homotopic.*

(ii) *If G is connected, then f is null homotopic if and only if $f \mid BT \simeq *$; or equivalently if and only if $\tilde{H}^*(f; \mathbf{Q}) = 0$.*

(iii) *$e_0: B\Gamma \to \mathrm{map}(BG, B\Gamma)_0$ induces an isomorphism of cohomology with arbitrary finite coefficients.*

Theorem 5 follows from the vanishing—for any G and p—of higher limits of constant p-local functors over $\mathscr{R}_p(G)$. It is closely related to a theorem of Zabrodsky [15].

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF WARSAW, PL00901 WARSAW, POLAND

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY 40506-0027

DEPARTMENT OF MATHEMATICS, AARHUS UNIVERSITY, AARHUS, DENMARK