background material is included, too. Many simple proofs are given. It is an excellent source of ideas for a lecture to a mathematics club or high school.

My main criticism of it is the poor quality of typesetting. It is far inferior to the TEX to which I have become accustomed. The 101-pp. bibliography has a verbose three-column format. One column holds the year of the publication and another gives the author. The title and the other information occupies the third column.

The book is so popular that the first edition sold out completely in only one year. A second edition will appear soon. Many of the records have been broken; these will be updated in the new edition. (Some records in this review will be superseded before it appears in print.) A few typographical errors will be corrected as well, but, alas, the book will not be retypeset.

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Limit theorems for stochastic processes, by J. Jacod and A. N. Shiryaev. Grundlehren der Mathematischen Wissenschaften, vol. 288, Springer-Verlag, Berlin, Heidelberg, New York, 1987, xvii + 600 pp., $98.00. ISBN 3-540-17882-1

1. Introduction. Two important threads in the fabric of stochastic processes come together in this monograph: semimartingales and convergence in distribution (weak convergence) of stochastic processes. Even though semimartingales arise mainly in the context of stochastic integration, they are describable, via their so-called predictable characteristics, in a manner that makes them the natural class of stochastic processes for which convergence in distribution is sufficiently tractable to admit a broad theory yet still specific enough to be interesting.

Before delving more deeply into its contents and significance, we consider separately the two threads the book joins.

2. Semimartingales. Arguably the two most fundamental continuous time stochastic processes are the Brownian motion (W_t) and the Poisson process (N_t). They share properties of

- *Independent increments*: increments in either process over disjoint time intervals are independent random variables,
- *Stationary increments*: the distribution of the increment of either process over an interval $(t, t+s]$ depends on s but not t,

but differ in another sense as much as possible:

- Brownian sample paths $t \to W_t$ are continuous (although, with probability one, nowhere differentiable),
- Poisson sample paths $t \to N_t$ are integer-valued step functions, all of whose jumps are of size $+1$.

Both processes, moreover, admit nearly identical "martingale characterizations." To explain, recall that a process X is a martingale with respect to σ-algebras $\mathscr{H}_t$, with $\mathscr{H}_t$ representing the observed history over $[0, t]$, if for each t and s,

$$E[X_{t+s}|\mathscr{H}_t] = X_t. \tag{2.1}$$

Equivalently, provided one can make sense of the (forward) differential dX_t,

$$E[dX_t|\mathscr{H}_{t-}] = 0 \tag{2.2}$$

for each t, where $\mathscr{H}_{t-}$ corresponds to the history over $[0, t)$.

The two processes are then characterized as follows:

- Both (W_t) and $(W_t^2 - t)$ are martingales:

$$E[dW_t|\mathscr{F}_{t-}^W] = 0 \tag{2.3}$$

and

$$E[(dW_t)^2|F_{t-}^W] = dt. \tag{2.4}$$

Here $\mathscr{F}_{t-}^W = \sigma(W_u\colon u < t)$. Together with continuity, (2.3) and (2.4) characterize Brownian motion.

- The processes $(N_t - t)$ and $((N_t - t)^2 - t)$ are martingales:

$$E[dN_t - dt|\mathscr{F}_{t-}^N] = 0 \tag{2.5}$$

and

$$E[(dN_t - dt)^2|\mathscr{F}_{t-}^N] = dt. \tag{2.6}$$

These, along with the "all jumps of size 1" condition, characterize the Poisson process.

Note in addition the analogies between (2.3) and (2.5) and between (2.4) and (2.6).

The Brownian motion and Poisson process are extreme points of the class of stochastic processes with independent increments. For such a process X, provided $X_0 = 0$ and there are no fixed times of discontinuity, then for each t,

$$\log E[e^{iuX_t}] = iub_t - \frac{u^2}{2}c_t + \int \left(e^{iux} - 1 - iuh(x)\right) F_t(dx), \tag{2.7}$$

where b_t, c_t are real, F_t is a positive measure with $\int \min\{1, x^2\} F_t(dx) < \infty$, and h is a bounded function such that $h(x) = x$ near the origin. To interpret, a process with independent increments is, heuristically, a convex combination of a deterministic motion, represented by the drift function b_t; a continuous Gaussian component (a nonstationary version of Brownian motion) with variance function c_t; and a jump component with the property that the point process whose points are pairs consisting of times and sizes of jumps is a (two-dimensional) Poisson random measure.

Moreover, with $\psi_t(u)$ denoting the function defined by the right-hand side of (2.7), for each u,

$$(e^{iuX_t}/e^{\psi_t(u)})_{t\geq 0} \text{ is a martingale.} \tag{2.8}$$

The essence of a semimartingale is to replace the deterministic elements in (2.7) by predictable stochastic processes (B_t), (C_t) and $(\nu([0,t] \times dx))$ in such a manner that (2.8) remains valid. (Note that $(\psi_t(u))$ now becomes random as well.) There is, however, a more basic definition: a process X is a *semimartingale* if it admits a decomposition

$$X = X_0 + M + A, \tag{2.9}$$

where M is a local martingale and A is a process with sample paths of locally bounded variation; X is termed *special* if A is predictable. The theoretical significance of semimartingales is that they are the only "integrators" for which a reasonable theory of stochastic integration can be constructed.

3. Convergence in distribution. Convergence in distribution of random variables, typified by the central limit theorem, dates, of course, almost from the antiquity of probability theory, and convergence of finite-dimensional distributions of stochastic processes is a generally straightforward modification of it. The latter is inadequate, however, to ensure convergence of many functionals of stochastic processes of interest, and one requires the stronger notion of convergence in distribution for stochastic processes, which entails viewing processes as random elements of a function space endowed with a particular topology, and their laws as probability measures on this space.

The "classical method" (cf. Billingsley, 1968) for establishing that a sequence (X^n) of stochastic processes (whose sample paths lie in the Skorohod space $D[0,1]$ of functions that are right-continuous with left-hand limits) converges in distribution to a process X, denoted by $X^n \xrightarrow{d} X$, is to prove that

- (X^n) is tight, a property equivalent to relative compactness, but easier to verify
- The finite-dimensional distributions of X^n converge to those of X
- X is characterized by its finite-dimensional distributions.

Of these, the third is typically trivial; the first is unavoidable, and albeit sometimes difficult, tractable in a variety of situations. The second, by contrast, is often essentially impossible unless the limit has independent increments; the reason is that many descriptions of stochastic processes, notably via predictable characteristics, provide no useful information about the finite-dimensional distributions.

In this monograph the authors explore in detail the "martingale method" paradigm of demonstrating convergence in distribution by proving that

- (X^n) is tight
- The characteristics of X^n converge to those of X
- X is characterized by its characteristics.

In this latter setting the difficult step is the third rather than the second.

4. The book. The book is organized into ten chapters, whose contents we summarize.

I. *The general theory of stochastic processes, semimartingales and stochastic integrals.* This is standard material on the "théorie générale du processus," available from many sources.

II. *Characteristics of semimartingales and processes with independent increments.* Here the key concepts are introduced. Given a semimartingale X, and a truncation function h (h is bounded with compact support and $h(x) = x$ in a neighborhood of the origin), one introduces processes

$$\mu^X = \sum_s 1(\Delta X_s \neq 0)\varepsilon_{(s,\Delta X_s)}$$

$$\breve{X}(h)_t = \sum_{s \leq t}[\Delta X_s - h(\Delta X_s)]$$

$$X(h) = X - \breve{X}(h),$$

where $\Delta X_s = X_s - X_{s-}$ is the jump of X at time s. These represent, respectively, the times and sizes of the jumps of X, the "large" jumps (for ΔX_s sufficiently small, $h(\Delta X_s) = \Delta X_s$) and X with the large jumps removed. The process $X(h)$ is a special semimartingale and from its canonical decomposition

$$X(h) = X_0 + M(h) + B(h), \tag{4.1}$$

one defines the triplet of characteristics of X, consisting of

- The predictable process B in (4.1)
- The quadratic variation process C of the continuous martingale part of X
- The compensator ν of the "measure of jumps" μ^X.

Physically, they generalize the drift, variance of the Gaussian part and Lévy measure (cf. (2.7)) of a process with independent increments by permitting these objects to depend in a predictable manner on the (strict) past of the process.

It is these characteristics whose convergence implies that of semimartingales. Somewhat more precisely, one deals rather than C with the *modified second characteristic* $\tilde{C}$, the quadratic variation of the martingale $M(h)$ appearing in (4.1).

III. *Martingale problems and changes of measures.* Here the issue is whether and to what extent the characteristics of a semimartingale in fact do characterize its distribution. Though this is not so in general, fortunately it is true (and all that one needs) for processes that typically arise as limits. The approach is via martingale problems, the most fundamental of which is the following: given a process X; a probability P on an initial σ-algebra $\mathscr{H}_0$; and a predictable process B, a continuous process C, and a predictable random measure ν, does there exist a probability P^* extending P with respect to which X is a semimartingale with characteristics (B, C, ν)? This question is answered affirmatively in key special cases.

Elsewhere in the chapter the authors discuss martingale representation theorems, absolutely continuous changes of probability law, and Girsanov theorems, which give likelihood ratios explicitly.

IV. *Hellinger processes, absolute continuity and singularity of measures.* The main issue addressed here is absolute continuity and singularity of probability laws of semimartingales, to which are applied Hellinger distances, integrals and processes.

V. *Contiguity, entire separation, convergence in variation.* Contiguity and entire separation are, roughly speaking, absolute continuity and singularity in the limit for two sequences of probability measures. They have various statistical and mathematical implications, none of which, regrettably, is explored here. (Some are described in Greenwood/Shiryaev, 1985.) Hellinger processes are the main tool. Using bounds on variation distances between measures in terms (essentially) of $L^{1/2}$-norms of Hellinger processes, several results on convergence in variation for stochastic processes, which is much stronger than convergence in distribution, are derived.

VI. *Skorokhod topology and convergence of processes.* Like Chapter I, this is largely standard material; there are, however, several criteria for tightness tailored to semimartingales.

VII. *Convergence of processes with independent increments.*

VIII. *Convergence to a process with independent increments*

IX. *Convergence to a semimartingale*

Finally, in Chapters VII-IX appear the limit theorems engendering the book's title. The general goal of providing conditions implying convergence of a sequence of semimartingales to a limit semimartingale is approached incrementally: first all processes are assumed to have independent increments, then only the limit, then none. The questions posed are of one prototypical form: given semimartingales X^n with modified characteristics $(B^n, \tilde{C}^n, \nu^n)$ and a semimartingale X with characteristics

$(B, \tilde{C}, \nu)$, derive necessary and sufficient conditions for $X^n \xrightarrow{d} X$ in terms of convergence of $(B^n, \tilde{C}^n, \nu^n)$ to $(B, \tilde{C}, \nu)$. Most of these results are too complicated to reproduce here, but in general terms the convergence required of characteristics is

$$\sup_{s\le t} |B_s^n - B_s \circ X^n| \xrightarrow{P} 0, \qquad t \ge 0$$

$$\sup_{s\le t} |\tilde{C}_s^n - \tilde{C}_s \circ X_n| \xrightarrow{P} 0, \quad t \ge 0 \tag{4.2}$$

$$\sup_{s\le t} |g * \nu_s^n - (g * \nu_s) \circ X^n| \xrightarrow{P} 0, \qquad t \ge 0.$$

X. *Limit theorems, density processes and contiguity.* Here the authors explore some contiguity-related consequences of their limit theorems.

The main text is complemented by bibliographical comments, a list of 254 references and indices of symbols and terminology.

5. Comments. Pascal is reputed to have apologized to a friend for "writing such a long letter, but I did not have time to write a short one." This is not a short book. Jacod and Shiryaev appear to have exercised no selectivity, and in consequence have produced an almost impenetrable morass of highly technical theorems, with little interpretation or explanation. The unremitting ponderousness of the book is relieved only (but this is significant!) their repeatedly working out in detail examples and special cases, such as the general theory of discrete time stochastic processes, and refined theorems corresponding to limits that are point processes (Poisson limit theorems, in particular), Gaussian martingales (central limit theorems) and martingales.

Admittedly this is very difficult material, but nevertheless graceful, thoughtful exposition (Blumenthal/Getoor, 1968, remains a premier illustration) can elucidate even the most complicated theory. That the English is considerably less than fluent compounds the problem.

Few readers will have the endurance to reach the main results by starting at the beginning. Indeed, an amusing misprint, on p. 499, to the effect that "We at least [*sic*] proceed to heart of our subject" seems to summarize the situation perfectly.

This is a pity, because there is a wealth of material here that is useful and beautiful, and is otherwise available only in journals, but the authors have rendered most of it, to use a relevant technical phrase, totally inaccessible. I anticipate that the book will be referred to most often in terms such as "It's probably in Jacod and Shiryaev, but ...."

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