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Convex bodies and algebraic geometry: An introduction to toric varieties, by Tadao Oda. Ergebnisse der Mathematik und ihrer Grenzgebiete 3, Folge, Band 15, Springer-Verlag, Berlin, Heidelberg, New York, 1988, \$78.00. ISBN 3-540-17600-4

The reader has heard the cliché that algebraic varieties are the locuses of solutions of polynomial equations; true as far it goes, but even in simple cases, one can know a lot about the equations and almost nothing about the solution set. Conversely, and more to the present point, even for a variety having an extremely natural and simple description, writing out the defining equations might be enormously expensive and unrewarding.

First of all, I want to give the flavour of toric geometry with two simple examples illustrating the main point, before discussing the background and the content of Professor Oda's very substantial book. Consider the quotient $\mathbf{C}^n/G$ of $\mathbf{C}^n$ by a diagonalised group action

$$(x_1, \dots, x_n) \mapsto (\varepsilon_1(g)x_1, \dots, \varepsilon_n(g)x_n),$$

where G is a finite Abelian group and $\varepsilon_i \colon G \to \mathbf{C}^*$ characters of G. This quotient can be seen as an explicit affine variety: make a list of G- invariant monomials, that is,

$$\left\{ x^{\mathbf{m}} = \prod x_i^{m_i} \,\middle|\, m_i \geq 0 \text{ and } \prod \varepsilon_i(g)^{m_i} = 1 \forall g \in G \right\},$$

then write out all the multiplicative relations between the generators, and finally, take these as the defining equations of a variety. Try it with

$$n = 2, \qquad G = \mathbf{Z}/22 \quad \text{and} \quad (x, y) \mapsto (\zeta x, \zeta^9 y)$$

where $\zeta = \exp(2\pi i/22)$. As a second example, if $0 < a < n-1$, the 2×2 minors expressing the determinantal condition

$$\operatorname{rk}\begin{bmatrix} x_0 & x_1 & \cdots & x_{a-1} & x_{a+1} & \cdots & x_{n-1} \\ x_1 & x_2 & \cdots & x_a & x_{a+2} & \cdots & x_n \end{bmatrix} \le 1$$

are homogeneous monomial equations, and define the *rational normal surface scrolls* $\mathbf{F}_b \subset \mathbf{P}^n$ (with $b = |n-2a|$). These surfaces have appeared throughout projective and algebraic geometry since their introduction by C. Segre **[0]** and del Pezzo in the 19th century. (It's a common postwar provincialism to refer to them as *Hirzebruch surfaces*.)

The varieties in either example have lots of nice simple structural properties which are only obscured by writing out tables of generators and defining equations; thus a coordinate hyperplane $(x_i = 0) \subset \mathbf{C}^n$ drops to a codimension 1 locus $D_i \subset \mathbf{C}^n/G$ in the quotient, but to see the ideal of D_i one has the job of listing all G-invariant monomial multiples of x_i.

Toric varieties or *torus embeddings* is a class of algebraic varieties obtained by abstracting out the key monomial structure possessed by these examples; these varieties occur just about everywhere in math, and they are to general algebraic varieties much as Abelian groups are to general groups. The main point of toric geometry is that any reasonable question concerning toric varieties can be phrased in terms of arrangements of convex bodies in lattices; this leads to a dictionary between the algebraic geometry of toric varieties and the convex geometry of cones $\sigma \subset \mathbf{Z}^n$ (I intend to be sloppy: the σ are polyhedral cones in $\mathbf{R}^n$, with vertexes in $\mathbf{Z}^n$).

For example, the scroll $\mathbf{F}_b$ is a union of 4 affine pieces, each isomorphic to $\mathbf{C}^2$, glued together by birational maps of the form $(x,y) \mapsto (x^a y^b, x^c y^d)$. The figure gives a diagram in $\mathbf{Z}^2$ from which the trained eye can read off at once all the geometric properties of the scroll $\mathbf{F}_b$. The 4 affine pieces of $\mathbf{F}_b$ are given by the layout of 4 *cones* forming a *fan*; the 6 matrixes taking a basis of $\mathbf{Z}^2$ associated with one cone to another define the glueing maps. The fact that the 4 cones cover all of $\mathbf{R}^2$ means that $\mathbf{F}_b$ is compact. The vertical projection of $\mathbf{Z}^2$ (compatible with the fan in an obvious sense) gives the $\mathbf{P}^1$-fibration of $\mathbf{F}_b$. One section of $\mathbf{F}_b$ has negative selfintersection (normal bundle) because the union of the top two cones is convex; amalgamating them into a single cone contracts the section to a point, etc.

I now discuss sample areas of math where toric geometry plays an important role, without trying to sort out the historical issue of which were the original motivation, and which have subsequently seen to be closely related.

By the resolution of singularities, many problems in algebraic geometry reduce to a normal crossing divisor in a complex manifold. Locally, this is a union of coordinate hyperplanes, say D: $(x_1 \cdot \ldots \cdot x_k = 0) \subset U = \mathbf{C}^n$. A finite covering $V \to U$ branched along U is locally given by taking various roots of monomials, say $\sqrt[r_i]{(x_1^{a_1} \cdot \ldots \cdot x_k^{a_k}}$. I can take V to be normal, and then it's of the form $\mathbf{C}^n/G$ as in my first example. How best to resolve the singularities of V is a question that goes back to F. Klein, and more especially to R. J. Walker's original proof of the resolution of

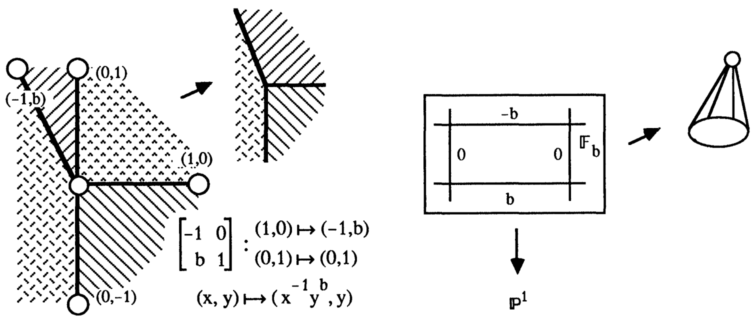

FIGURE 1

surface singularities. In modern terms, the answer in the surface case is to make a fan in $\mathbf{Z}^2$ given by a finite continued fraction; the resolution is the associated toric variety. If $f: X \to \mathbf{C}$ is an analytic function on a variety, Hironaka's resolution of singularities allows us to blow up $f^{-1}(0)$ to be a normal crossing divisor $D \subset U$, so locally $f = x_1^{a_1} \cdot \ldots \cdot x_k^{a_k}$. Mumford's semistable degeneration theorem says that for some $r \geq 1$, the covering $V \to U$ associated with a suitable root $\varphi = \sqrt[r]{f}$ has a toric resolution such that $\varphi^{-1}(0)$ is a normal crossing divisor, and locally $\varphi = x_1 \cdot \ldots \cdot x_k$; that is, one can reduce to the case when all the $a_i = 1$. Mumford's seminar **[5]** was responsible for promoting toric geometry as a subject in its own right in the early 1970s (and also for some of the awful terminology).

Another way of describing a toric variety is as a partial compactification of an algebraic torus, or a *torus embedding:* an n-dimensional toric variety X contains an algebraic torus $\mathbf{T} = \mathbf{C}^* \times \cdots \times \mathbf{C}^*$ (n factors), with an action of $\mathbf{T}$ on X extending the multiplication map $\mathbf{T} \times \mathbf{T} \to \mathbf{T}$. A monomial x^μ on X is then an eigenvector of the action of $\mathbf{T}$ on the rational function field of X for some character $\mu: \mathbf{T} \to \mathbf{C}^*$; if X is affine, the monomials that are regular on X form a convex cone in the character lattice of $\mathbf{T}$, and dually, X can be described as the space of representations of this semigroup. Many of the initial definitions and results of toric geometry first occur in Demazure's study of maximal connected algebraic subgroups of the Cremona group **[2]**. The main idea here is that the automorphism group of a toric variety X is an algebraic group G with $\mathbf{T} \subset X$ as its maximal torus, and the cones in the character lattice of $\mathbf{T}$ used to construct X also describe the root system of G.

Quotients of symmetric domains by arithmetic groups appear in the study of modular forms in arithmetic and analysis, and in the geometry of moduli spaces; the question of compactifying these is another important application of toric geometry. If a polyhedral fundamental domain for the quotient has a tube going out to a cusp at infinity, a compactification is a piece of a complex space that caps this off; according to Satake, this can often be described in terms of glueing together open sets in toruses $(\mathbf{C}^*)^n$ by monomial identification. The systematic study of these compactifications leads to problems of subdivisions of cones in number fields. The most beautiful example of this is Hirzebruch's resolution [3] of the cusps of Hilbert modular surfaces: take the periodic continued fraction of a real quadratic irrationality $\sqrt{D}$, use it to make a fan in $\mathbf{R}^2$ and hence a toric variety, and finally glue this to itself using a monomial identification coming from a unit of the number field $\mathbf{Q}(\sqrt{D})$.

The toric dictionary can also be read in the opposite direction, associating with purely combinatorial data the living structure of an algebraic variety. This can be viewed as a far-reaching extension of the geometric realisation of a simplicial complex. For example, Riemann-Roch and Serre duality for complex projective varieties apply directly to questions on the number of lattice points in the interior of a convex cone, and the Hodge index theorem to the isoperimetric inequalities for plane polygons. Via the Stanley-Reisner ring (the coordinate ring $k[V]$ of $V \subset k^n$, where V is a union of coordinate linear subspaces), cohomological properties of varieties translate into new properties of combinatoric objects, such as Cohen-Macaulayness. The most spectacular result in this vein is Stanley's paper [4], which proves the necessity of a criterion for a sequence of integers to be the number of faces of an r-dimensional simplicial convex polytope P. This goes from P to a complex projective toric variety X_P, and then to its cohomology ring $H^*(X_P, \mathbf{Z})$, a surprising new object in the study of polytopes; the key idea of the proof is then to use inequalities on the Betti numbers of X_P coming from the hard Lefschetz theorem, a deep result in the Hodge theory of projective varieties.

Already in two of the applications discussed above, toric varieties have appeared as local analytic models for varieties or complex spaces, thus overcoming their somewhat limited scope for self-expression. Toric varieties also commonly occur as ambient spaces, with $\mathbf{A}^n$ and $\mathbf{P}^n$ only the most primitive examples. It often happens that interesting properties of a polynomial function $f(x_1, \dots, x_n) = \sum a_{\mathbf{m}} x^{\mathbf{m}}$ or the variety $(f = 0)$ depend in the first instance not on the actual coefficients $a_{\mathbf{m}}$, but only on whether $a_{\mathbf{m}}$ is zero. The *Newton polyhedron* of f is defined as the convex hull in $\mathbf{Z}^n$ of the monomials $x^{\mathbf{m}}$ with $a_{\mathbf{m}} \neq 0$, and toric varieties associated with it provide natural ambient spaces for the study of f; by results of Kushnirenko, Khovansky, Varchenko and others, almost all properties of f of interest in singularity theory and algebraic geometry can be phrased in terms of Newton(f), provided that f is *nondegenerate*. This is an enormous extension of the toric dictionary to a large class of subvarieties of toric varieties.

This book is of the 'Topics' kind, and is remarkable for the amount of material covered in its 200 pages. This divides into foundational stuff on toric varieties, applications to the geometry of surfaces and 3-folds, and an appendix containing preliminaries on convex geometry, as well as a treatment of Stanley's proof. Quite a lot appears here for the first time in book form, in particular the material on toric 3-folds and birational geometry, and results of the Sendai school: the detailed treatment of M-N. Ishida's dualising complex for toric varieties, and the material in §4.2 on compactifications and Tsuchihashi cusps. The book does not cover Mumford's semistability theorem, the material on Newton polyhedrons, or arithmetic aspects of the compactification of cusps.

The chapters are uneven in level of difficulty, and several of the sections start with a rather hard technical treatment of foundational material, before going on to discuss quite simple examples; the reader might for example get a lot out of the first half of §1.5, the example on pp. 108–109, §4.1, or the final §A.5 after only a brief dip into the definitions in §1.1. Although familiar with the material and reasonably competent in algebraic geometry, the reviewer gave up trying to decode the proofs of compactness on pp. 16–17 and p. 21. §3.2 has the technical aim of giving the dualising complex in explicit form, and some readers may find the sections on differential forms in [**1**] easier going.

The book is a line by line translation of the Kinokuniya Japanese edition; the original was possibly intended for the use of graduate students with a higher technical stress tolerance than their western counterparts. The formal language of toric geometry, designed for stating and checking the truth of theorems in reasonable generality, is almost as painful and unnatural to write out as it is to decipher, whereas the subject matter itself is really very easy. To my knowledge, anyone who has seriously got into toric geometry has largely bypassed the formalism, using the experience of practical computation with a handful of concrete examples as the substantive justification for the truth of results. My only serious reservation with the book is that the author has not been able to pass on this experience to the reader; he might have done well to trespass for another 50 pages on the goodwill of the Ergebnisse editors by providing each section with exercises and worked numerical examples.

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