BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 21, Number 2, October 1989

0273-0979/89 $1.00 + $.25 per page

Subnormal subgroups of groups, by John C. Lennox and Stewart S. Stonehewer. Oxford Mathematical Monographs, Clarendon Press, Oxford, 1987, xiii + 253 pp., $69.00. ISBN 0-19-853552x

Normality is not a transitive relation for subgroups of a group; the transitive closure of this relation yields subnormality: The subgroup H of the group G is called *subnormal in* G if there exists a finite chain $\{H_i\}$ of subgroups H_i of G with $H = H_0 \leq H_1 \leq \cdots \leq H_i \leq H_{i+1} \leq \cdots \leq H_n = G$ such that H_i is a normal subgroup of H_{i+1} for every $i = 0, \ldots, n-1$. The length of a minimal such chain is the *subnormal defect* of H in G.—In finite groups G the subnormal subgroups are just those subgroups through which passes some composition chain of G; and the Jordan-Hölder theorem states that the set of factors H_{i+1}/H_i of a composition chain of G does not depend on the composition chain chosen. So subnormal subgroups somehow reflect the "normal structure" of the group G. It is almost obvious that the intersection of any two subnormal subgroups of a group G is also subnormal in G. What about the join, i.e. the subgroup of G generated by these two subnormal subgroups? The answer is simple, if the subnormal subgroups H, K are so embedded in G that they permute, i.e., $HK = KH = \langle H, K\rangle$, in this case the subgroup HK is subnormal in G. This observation motivates the search for permutable (subnormal) subgroups of a group. Picking up the above question of Remak's for groups with a (finite) composition chain, in particular for finite groups, H. Wielandt proved in 1939 that the join of two subnormal subgroups is again subnormal. So in finite groups the subnormal subgroups form a sublattice of the lattice of all subgroups. That this is not so for groups in general was first pointed out by Zassenhaus; further examples were produced by P. Hall, Derek Robinson, J. Roseblade and St. Stonehewer....

Wielandt proved in his work that, under his restrictions, a perfect subnormal subgroup H, i.e. one which is generated by the set of its commutators, permutes with every other subnormal subgroup. So the inner structure of H reflects on the way H is embedded in G. J. Roseblade found in 1964 another proof of this result and a sweeping generalisation: If H and K are any two groups, consider their maximal abelian factor groups H/H', K/K' and their tensor product (over $\mathbf{Z}$) $H/H' \otimes K/K'$. These groups are called *orthogonal* (to each other) if $H/H' \otimes K/K' = (0)$. In particular, if H is perfect, then H/H' is trivial and H is orthogonal to every group. Roseblade showed that if two subnormal subgroups H and K of a group G are orthogonal, then they permute and hence their join HK is subnormal in G. Roseblade also gave a converse to this result: If the groups H and K are not orthogonal, then there is a group G with subnormal subgroups isomorphic to H and K which do not permute.

These two results set the themes for much subsequent research: What structural properties of a group G ensure that the join of any two subnormal subgroups of G is again subnormal? Which inner properties of the subnormal subgroups H and K of the group G ensure that $\langle H, K\rangle$ is again subnormal?—A spectacular result in this second direction, generalising Roseblade's result, is due to J. P. Williams (1982): If the tensor product $H/H' \otimes K/K'$ of the groups H and K has finite rank over its largest divisible torsion subgroup, then in any group G any two subnormal subgroups isomorphic to H and K have subnormal join. Conversely, if the tensor product $H/H' \otimes K/K'$ does not have this "smallness" property, then the groups H and K may be embedded into some group G as subnormal subgroups so that their join in G is not subnormal.

The book under review discusses most of the known results on subnormal subgroups in a very clear and pleasant fashion, very much with the prospective reader in mind. Some of the key results are presented with more than one proof; often the later result and its proof generalise the earlier one. The questions mentioned are generalised to arbitrary sets of subnormal subgroups of a group G. Another type of question is taken up: How can one recognise whether a subgroup H of a group G is subnormal? Here it seems some rather strong finiteness conditions are needed, or one must generalise the notion of subnormal subgroup to that of *serial* subgroup where Hickin and Phillips have given a local characterisation. But then a simple group may contain a proper serial subgroup: in a linearly ordered group every convex subgroup is serial. Hartley observed 1971 that in locally finite groups the serial subgroups form a complete sublattice of the subgroup lattice.

If in the finite group G all subgroups are subnormal, then G is nilpotent by an old result of Wielandt.[1] Heineken and Mohamed gave an example of a group with trivial centre in which every subgroup is subnormal. Quite recently W. Möhres proved that without any finiteness assumptions on the group G, the condition that every subgroup of G be subnormal forces G to be soluble.... On pp. 219/220 Lennox and Stonehewer mention a problem on finite groups which has intrigued me very much over the years: Suppose S is a subgroup of the finite group G such that for every prime p and every Sylow p-subgroup P the intersection $P \cap S$ is a Sylow p-subgroup of S. Is S subnormal in G? It is not difficult to reduce this problem to the case where the groups G and S both are nonabelian simple. Using the classification of the finite simple groups and knowledge of their subgroup structure Peter Kleidman has recently given an affirmative answer to this question.

Asking for the behaviour of subnormal subgroups of a group G seems at the first glance asking for the workings of some formal machinery; the Jordan-Hölder theorem appears in this light. But indeed, one is dragged deeply into the inner structure of the group G and has to consider the structure of the subnormals as well as their embedding into G. Thus this subject matter seems ideally suited for a graduate course on (infinite) group theory, leading from formal considerations to structural insight. This book is an ideal text for such a course; the many results mentioned without proofs in the text can serve as informative (and demanding) exercises. A beautiful invitation to group theory!

O. H. Kegel
University of Freiburg

[1] This was generalised by Roseblade to infinite groups in which all subgroups are subnormal with bounded defect.