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ON SOLVABLE SUBGROUPS OF THE SYMMETRIC GROUP

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1. Introduction.. In this note we give exact values of certain invariants of the symmetric group S_n of degree n.

Let n be a positive integer, p a prime, $\sigma(G)$ the derived length and $\nu(G)$ the nilpotent length of a solvable group G. Let $\mathrm{SOLV}(n)$ denote the set of all solvable subgroups of S_n and put

$$\begin{aligned} \mathrm{SOLV}(n,p') &= \{G \in \mathrm{SOLV}(n) | p \nmid |G|\}, \\ \sigma(n) &= \max\{\sigma(G) | G \in \mathrm{SOLV}(n)\}, \\ \nu(n) &= \max\{\nu(G) | G \in \mathrm{SOLV}(n)\}. \end{aligned}$$

Similarly one defines $\sigma(n,p')$ and $\nu(n,p')$.

Let $\mathbf{N}$ be the set of all nonnegative integers. For $t \in \mathbf{N}$ we put $s(t) = \min\{m \in \mathbf{N} | \sigma(m) = t\}$ and $n(t) = \min\{m \in \mathbf{N} | \nu(m) = t\}$. For a partial ordered set L we denote by μL the set of all maximal elements in L. We put $\Sigma(t) = \{G \in \mu\,\mathrm{SOLV}(s(t)) | \sigma(G) = t\}$ and $\Sigma(t,p') = \{G \in \mu\,\mathrm{SOLV}(s(t,p'),p') | \sigma(G) = t\}$. Similarly one defines $N(t)$ and $N(t,p')$.

We define the structure of all elements of the sets $\Sigma(t)$, $\Sigma(t,p')$, $N(t)$ and $N(t,p')$.

We assume that, as permutations groups, S_m has degree m, $\mathrm{AGL}(2,3)$ has degree 9, the cyclic group $C(p)$ of order p has degree p, the groups $\mathrm{AGL}(1,p)$ and $\frac{1}{2}\mathrm{AGL}(1,p)$ (=the subgroup of index 2 in $\mathrm{AGL}(1,p)$) have degree p.

We say that a group W is *of type* $\{B_1^{k_1}, \dots, B_s^{k_s}\}$ if W a wreath product of k_1 copies of the permutation group B_1, k_2 copies of the permutation group B_2 and so on (the order of the factors is arbitrary).

2. Main results.

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THEOREM 1. *Let $G_t \in \Sigma(t)$. If $t < 4$, then $G_t = S_{t+1}$. If $t = 4$, then G_t is of type $\{S_2, S_4\}$. Suppose now that $t > 4$.*

(a) G_{5k} *is of type* $\{\mathrm{AGL}(2,3)^k\}$ (*so* $s(5k) = 9^k$).
(b) G_{5k+1} *is of type* $\{S_4^2, \mathrm{AGL}(2,3)^{k-1}\}$.
(c) G_{5k+2} *is of type* $\{S_3, \mathrm{AGL}(2,3)^k\}$.
(d) G_{5k+3} *is of type* $\{S_4, \mathrm{AGL}(2,3)^k\}$.
(e) G_{5k+4} *is of type* $\{S_4^3, \mathrm{AGL}(2,3)^{k-1}\}$ *and* $s(5k+4) = 4^3 \cdot 9^{k-1}$.

If the function s is known, one can restore σ.

THEOREM 2. *Let $G_t \in \Sigma(t, 2')$. Then*

(a) G_{2k} *is of type* $\{\frac{1}{2}\,\mathrm{AGL}(1,7)^k\}$ *and* $s(2k, 2') = 7^k$.
(b) G_{2k+1} *is of type* $\{C(3), \frac{1}{2}\,\mathrm{AGL}(1,7)^k\}$ *and* $s(2k+1, 2') = 3 \cdot 7^k$.

THEOREM 3. *If $G_t \in \Sigma(t, 3')$, then $G_t \in \mathrm{Syl}_2(S_{2^t})$.*

THEOREM 4. *If $p > 3$, then $\Sigma(t, p') = \Sigma(t)$.*

THEOREM 5. *Let $G_t \in N(t)$. Then $G_1 = S_2$. Suppose that $t > 1$.*

(a) G_{4k} *is of type* $\{\mathrm{AGL}(2,3)^a,\ S_3^{2(k-a)}\}$, $a \le k$.
(b) $G_{4k+1} = S_4 \operatorname{wr} H$ *where H is of type* $\{\mathrm{AGL}(2,3)^a,\ S_3^{2(k-a)-1}\}$, $a < k$.
(c) G_{4k+2} *is of type* $\{\mathrm{AGL}(2,3)^a,\ S_3^{2(k-a)+1}\}$, $a \le k$.
(d) $G_{4k+3} = S_4 \operatorname{wr} G_{4k}$ *and* $n(4k+3) = 4 \cdot 3^{2k}$.

THEOREM 6. *Let $G_t \in N(t, 2')$. Then*

(a) $G_{2k} \in \Sigma(2k, 2')$ *and* $N(2k, 2') = \Sigma(t, 2')$.
(b) $G_{2k+1} = C(3) \operatorname{wr} G_{2k} \in \Sigma(2k+1, 2')$ (*but* $N(2k+1, 2') \ne \Sigma(2k+1, 2')$).

THEOREM 7. *Let $G_t \in N(t, 3')$. Then*

(a) G_{2k} *is of type* $\{\mathrm{AGL}(1,5)^k\}$ *and* $n(2k, 3') = 5^k$.
(b) $G_{2k+1} = C(2) \operatorname{wr} G_{2k}$, $n(2k+1, 3') = 2 \cdot 5^k$.

THEOREM 8. *If $p > 3$, then $N(t, p') = N(t)$.*

If $G = p_1^{m_1} \cdots p_s^{m_s}$, then $\lambda(G) = m_1 + \cdots + m_s$. If G is solvable, its composition length $c(G)$ is equal to $\lambda(G)$. We put

$$c(n) = \max\{c(G) \mid G \in \mathrm{SOLV}(n)\}.$$

Similarly one defines $c(n, p')$.

THEOREM 9. *Let $G \in \mathrm{SOLV}(n)$ be transitive and $c(G) = c(n)$. Then*

(a) $n = 4^k$, *G is of type* $\{S_4^k\}$.
(b) $n = 2 \cdot 4^k$, $G = H \operatorname{wr} S_2$ *where H is from* (a).
(c) $n = 3 \cdot 4^k$, $G = H \operatorname{wr} S_3$ *where H is from* (a).
(d) $n = 6 \cdot 4^k$, $G = H \operatorname{wr} F$ *where H is from* (a), *F is of type* $\{S_2, S_3\}$.

THEOREM 10. *Let* $G \in \mathrm{SOLV}(n, 2')$ *be transitive and* $c(G) = c(n, 2')$. *Then*

(a) $n = 3^k$, $G \in \mathrm{Syl}_3(S_n)$.
(b) $n = 5 \cdot 3^k$, $G = H \,\mathrm{wr}\, C(5)$ *where* H *is from* (a).
(c) $n = 7 \cdot 3^k$, $G = H \,\mathrm{wr}\, \frac{1}{2}\,\mathrm{AGL}(1,7)$ *where* H *is from* (a).

THEOREM 11. *Let* $G \in \mathrm{SOLV}(n, 3')$ *be transitive and* $c(G) = c(n, 3')$. *Then*

(a) $n = 2^t$, $G \in \mathrm{Syl}_2(S_{2^t})$.
(b) $n = 5 \cdot 2^t$, $G = H \,\mathrm{wr}\, \mathrm{AGL}(1,5)$ *where* H *is from* (a).

THEOREM 12. *If* $p > 3$, *then* n *is the same as in Theorem* 9 *and* $c(n, p') = c(n)$ *for any transitive* $G \in \mathrm{SOLV}(n, p')$ *with* $c(G) = c(n, p')$; G *is the group from Theorem* 9.

We put

$$o(n) = \max\{|G| \,|\, G \in \mathrm{SOLV}(n)\}$$

and

$$o(n, p') = \max\{|G| \,|\, G \in \mathrm{SOLV}(n, p')\}.$$

THEOREM 13. *Let a transitive group* $G \in \mathrm{SOLV}(n)$ *has an order* $o(n)$. *Then*

(a) $n = 4^k$, G *is of type* $\{S_4^k\}$.
(b) $n = 2 \cdot 4^k$, $G = H \,\mathrm{wr}\, S_2$ *where* H *is from* (a).
(c) $n = 3 \cdot 4^k$, $G = H \,\mathrm{wr}\, S_3$ *where* H *is from* (a).
(d) $n = 2 \cdot 3 \cdot 4^k$, $G = H \,\mathrm{wr}\, S_3 \,\mathrm{wr}\, S_2$ *where* H *is from* (a).
(e) $n = 3^2 \cdot 4^k$, $G = H \,\mathrm{wr}\, S_3 \,\mathrm{wr}\, S_3$ *where* H *is from* (a).

THEOREM 14. *If* $p > 3$ *and a transitive group* $G \in \mathrm{SOLV}(n, p')$ *has the order* $o(n, p')$, *then* $|G| = o(n)$.

THEOREM 15. *Let a transitive group* $G = \mathrm{SOLV}(n, 2')$ *has the order* $o(n, 2')$. *Then*

(a) $n = 3^k$, $G = \mathrm{Syl}_3(S_n)$.
(b) $n = 5 \cdot 3^k$, $G = H \,\mathrm{wr}\, C(5)$ *where* H *is from* (a).
(c) $n = 7 \cdot 3^k$, $G = H \,\mathrm{wr}\, \frac{1}{2}\,\mathrm{AGL}(1,7)$ *where* H *is from* (a).

THEOREM 16. *Let a transitive group* $G \in \mathrm{SOLV}(n, 3')$ *has the order* $o(n, 3')$. *Then*

(a) $n = 2^k$, $G \in \mathrm{Syl}_2(S_n)$.
(b) $n = 5 \cdot 2^k$, $G = \mathrm{AGL}(1,5) \,\mathrm{wr}\, H$ *where* H *is from* (a).

THEOREM 17. *Let* N *be a nilpotent subgroup of maximal order in* S_n. *If* $n \not\equiv 3 \pmod 4$, *then* $N \in \mathrm{Syl}_2(S_n)$. *If* $n \equiv 3 \pmod 4$, *then* $N = P \times C(3)$ *where* $P \in \mathrm{Syl}_2(S_{n-3})$.

THEOREM 18. *Let* N *be a nilpotent subgroup of maximal odd order in* S_n. *If* $n \not\equiv 5 \pmod 9$, *then* $N \in \mathrm{Syl}_3(S_n)$. *If* $n \equiv 5 \pmod 9$, *then* $N = P \times C(5)$ *where* $P \in \mathrm{Syl}_3(S_{n-5})$.

THEOREM 19. *Let N be a nilpotent subgroup of maximal p'-order in S_n, $p > 2$. If $p = 3$, then $N \in \mathrm{Syl}_2(S_n)$. If $p > 3$, then N be the group from Theorem* 17.

Other results in this direction are in my paper, *Subgroups of symmetric and alternating groups*, Izv. Severo-Kavkaz. Nauchn. Tsentra Vyssh. Shkoly, Estestvennye Nauki **1** (1981), 6–9.

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