Can one obtain the same applications, substituting h-transforms for Burdzy's excursion laws? Probably, especially since he uses h-transforms to construct the H^x in the first place. There is at least one reason not to do so though. His favourite technique is the "splicing together" of excursion laws in domains D_1, D_2 with $D_1 \triangle D_2$ small. This works well, as the transition mechanism is Brownian in both D_1 and D_2. A similar construction with h-transforms would be awkward and more technical. It is interesting to note that another σ-finite measure, due to Kuznetsov has recently provided analogous simplifications in the sometimes technical study of time-reversal. Perhaps excursion laws will prove to be as useful.

The book itself is nicely written and well organised. The existing literature consists of scattered articles by Burdzy and his colleagues, with various stages of evolution, both of hypotheses and results. The book sorts it all out very clearly and carefully. It provides little general background, instead referring to some of the sources we've listed. The book is photoreproduced, with no index, but the TEX is very readable.

In summary, excursion laws are interesting in themselves, yet arise naturally and have an elegant theory. As Burdzy's excellent treatise demonstrates, they may prove to be a valuable tool in analysis as well.

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Riemannian geometry, by S. Gallot, D. Hulin and J. Lafontaine. Universitext, Springer-Verlag, Berlin, Heidelberg, New York, 1987, xi + 248 pp., $29.00. ISBN 3-540-17923-2

"Of making many books there is no end." The intervening thousands of years have surely added the verification of experience to the Divine Authority behind that statement, and not least in mathematics. But for all the truly endless making of mathematics books, there remain gaps in the mathematical literature; there are still books one would like to see written. Many of these desiderata would be works at the intermediate graduate level. There is no shortage of elementary works, and whatever lack of efficacy one might perceive in the undergraduate and beginning graduate curricula is hardly attributable to an absence of textbooks. At the level of the professional, the publication of research results is so well oiled that access to new information is restricted far more by the reader's limitations of time and energy than by any shortage of things to read. But the easiest possible transition from the elementary to the research level is indeed often restricted by the absence of ideal reading material.

By way of example, we could recall the situation in algebraic geometry before the books of Griffiths and Harris, Hartshorne, Mumford, Shafarevitch, and other recent authors; earlier, learning the subject had depended on either extraordinary determination or apprenticeship to a master, or both. Similarly, in partial differential equations, before the works of Hörmander, Gilbarg and Trudinger, and others in the past couple of decades, one could only be au courant at Courant—or at some other institute devoted to the subject. Those fields have now had Boswells to their Johnsons, or, as it often happened, the Johnsons have done their own memorializing. But in Riemannian geometry the ideal introduction to the subject in its modern form has been slower to appear.

This is not to say that masterful works on Riemannian geometry have not been written. The point is rather that most of these have not been intended as introductory texts. The imposing survey by Kobayashi and Nomizu [5], covering the field virtually in toto up to 1969, functions ideally as an encyclopaedia, but its intentional generality of presentation of the basic material defies easy comprehension by the novice. (Volume II, which presents more specific albeit more advanced topics, is in fact surprisingly suitable for introductory purposes at that more advanced level.) Spivak's historical rhapsody [6], while a delight for a geometer to read, is too discursive and extended to serve ideally as a textbook; few students are tempted by a five-volume work. Klingenberg's magnum opus [4] could serve well as an introduction for a sufficiently mature student, but its unrelentingly profound treatment does not make the first steps easy, e.g., Hilbert manifolds ab initio. The lecture notes of Gromoll, Klingenberg and Meyer [2] are admirable, but have never been translated from the German. Cheeger and Ebin's elegant treatment [1] of geodesic geometry is too terse in its basic portions and too narrow in focus later to be a good general introduction, though it should be on every future geometer's reading list. And Hicks' appealing short introduction [3] is no longer in print, and in any case does not extend very far into the subject, though it would be hard to better its treatment of the topics it does cover.

It must be admitted at the outset that there may be an ineradicable difficulty in first encountering Riemannian geometry. A strict, formal definition of the basic object, the Riemannian curvature tensor, is of course easy. One can simply write down the formula, verify the correct behavior relative to coordinate changes, and consider the matter disposed of. But the inner meaning, the Innigkeit if you will, of curvature does not reveal itself so readily. Like the Rheingold, it lies buried in the depths. Reaching it requires rather less renunciation than Alberich had to make (otherwise there would be far fewer Riemannian geometers). But it does require some considerable time, effort and experience to see the protean forms of curvature each separately and simultaneously as a unified whole.

Curvature first forces itself upon one's attention as a commutator of differentiation operators, a viewpoint that leads immediately to the coordinate formula already alluded to. At the same time, it yields the natural second order "normal form" of the metric tensor's Taylor expansion. Out of this grows its relationship with local distance geometry. From this latter geometric view, curvature is just a concise description of the local behavior of geodesics. Specifically, for 2-manifolds (surfaces), one can follow Gauss and take as definition that the curvature at a point is the limit as $r \to 0$ of $(-3/\pi)r^{-3}(l(r) - 2\pi r)$ where $l(r) =$ the length of the circle of radius r around the given point. Some work is needed of course to see that the limit exists, but once that is done it becomes clear that smaller curvature is associated to larger $l(r)$ which in turn corresponds to faster (local) divergence of geodesics. This definition then extends to higher dimensions easily. One simply takes the sectional curvature of a 2-plane in the tangent space at a point to be the curvature of the (local) 2-manifold obtained by applying the geodesic exponential map to the 2-plane.

What all this amounts to is that, once the geodesics are determined as, say, locally minimizing curves, then the curvature ideas follow naturally, at least in principle. Pedagogically, however, the path is thorny. Intricate calculations, which seem darkly excessive to students grown accustomed to the endless summer sunshine of the world of Bourbaki, are needed to verify directly the existence of the relevant limits. Local invariant theory is nothing but Taylor series in principle, but the "nothing but" becomes formidable in practice. Over the years, increasingly slick methods of detouring around the computational boulders have been developed. And it is possible to present the basic definitions in an order and sequence that leads rapidly to the goal of a straightforward and conceptual definition of curvature. But the necessity of developing somehow an intuitive geometric understanding remains, as does the necessity of relating the definition to other appearances of curvature, e.g., curvature of submanifolds in terms of second fundamental form, curvature as the infinitesimal generator of holonomy, etc.

Such associated geometric intuitions and relationships must be assimilated before a student can begin to hear the themes of the symphonic development of Riemannian geometry, not to mention the following of the subsequent development itself. Clearly, the textbook writer who proposes to reveal clearly and comprehensively the fundamental concepts of geometry is faced with a demanding task. For the unwary, it could turn out as a contemporary proverb has it: "If you can keep your head when all about you are losing theirs, maybe you just don't understand the situation."

The present book is aimed directly at being an introduction to enable an intermediate-level graduate student to start toward professional competence in geometry. The authors have kept their heads, and shown their understanding of the situation in the face of the aforementioned proverbial difficulties of exposition by adopting a composite approach. Their essential method is what we have come to think of as French à la

Bourbaki, that is, formal, precise, terse and elegant in the French sense. But the terse abstraction is considerably leavened by careful, systematic attention to examples. Terse the book surely is. A three-page section entitled "Baby Lie Groups" covers the definition of Lie group and Lie algebra, the exponential map and the adjoint representation. (One is reminded of the *Punch* satire of the Reader's Digest Condensed Books where *War and Peace* begins on p. 28 and the next item begins on p. 29.) But examples are carefully presented in many cases, and the combined result is intended to move the student reasonably rapidly through the basic material while keeping some feeling of feet on the ground. The considerable number of exercises, with their solutions given, also offer the reader a helpful opportunity to acquire active experience with the subject.

The arrangement of the book is, in the first half, conventional. It proceeds through (I) manifold preliminaries (48 pp.); (II) Riemannian metrics, connections, and geodesics (53 pp.); and (part of) (III) curvature first and second variation, Jacobi fields and the Gauss Lemma, conjugate points and cut locus (29 pp.). This accounts for about 130 pp. of the book's total of 240. The treatment is based on covariant differentiation in the $D_X Y$ notation, rather than differential forms; this to my mind is a more direct, more easily comprehended approach than the Cartan structure equations, etc., involved with the forms. In particular, the covariant derivation method used makes a clear exposition possible in spite of the extremely rapid pace and condensed exposition.

Around p. 130, the book begins to accelerate even more, and to cover topics that are less systematically necessitated and more up to the authors' choice. Some of these choices seem unusual but desirable, e.g. the emphasis on Riemannian submersions.

Other choices seem simply idiosyncratic, for example, the absence of comparison results except in the volume case of Bishop, et. al. (One finds the name of Rauch only as a contribution to the circle of ideas leading to the sphere theorems; Toponogov's name occurs not at all.) In general, the book does not emphasize the theorems that arise from arc-length-variation geometry—e.g., the sphere theorems, the Cheeger-Gromoll-Meyer results on nonnegative curvature; indeed, it does not even mention the latter. The logic of this choice may be that these results have been well and carefully covered elsewhere ([**1**, **4**], for example). But the present book nevertheless presents a somewhat incomplete view in this regard, not really representative of geometry as a whole.

The second half of the book is so terse as to be in many cases almost a list of results, without even indication of proof. A section entitled "Curvature and Topology," which occupies four pages, covers pinching theorems, almost flat manifolds, and the finiteness and compactness theorems of Cheeger and Gromov. The section on the de Rham and Hodge Theorems sets what I am confident is an all-time speed record by disposing of the whole business in five pages flat. In short, the second half is in short reform from start to finish, resembling more Carl Lewis covering a hundred meters than a leisurely ramble through attractive countryside. It will surely function only as a combination of overview and guide to further reading, or as a suggestion list of further topics for the lecturer basing a course on the text. For the record, in addition to the topics already mentioned, the second half adumbrates: curvature and fundamental group (based on volume comparison results), Stokes' Theorem, spectral geometry and eigenvalue estimates, and curvature of submanifolds.

A reviewer of a book with pedagogical intent finds it incumbent to comment on the book's potential pedagogical effectiveness. In this regard, some additional comments come to mind. First, the introductory chapter on manifolds is too abbreviated to be useful except as review and establishment of notation. But, given that most American universities offer an independent preliminary "manifold theory" course, this will not be a problem in the usual setting wherein the book might be used. A second, more serious question arises about the terseness and formality of the presentation. It can be argued that the development of intuition and insight is an individual matter. And no doubt one person's illuminating details are another's prolix distraction. But, in my personal view, a more leisurely, less formalistic presentation would have made the book more compelling. The de-emphasis on byways, the apparent intention of proving things as quickly as possible has produced a somewhat fleshless, verging on deconstructionist view of geometry, or so it seems to me, although the solved exercises are a help in this regard.

The practicing geometer must develop intuition of some sort, computational, pictorial or some combination. It is impossible to imagine anyone successfully pursuing research in geometry in a truly formalistic way. Even the most formidably computational partial differential equations techniques in geometry are based on intuition as to how the computations might go—just taking the Laplacian of everything in sight is unlikely to yield much. And the most exotic applications of comparison theorems, such as Gromov's Betti number estimates are concatenations of directly pictorial constructions; to see the particular chain that leads to the result requires insight of such depth that it would be inconceivable in a formalist framework. But how to generate even a modicum of such insight for the reader of a book is a deep and subtle question, and the answer no doubt depends on the reader, too. This book presents its (first-half) material cleanly and precisely, but too many details and augmentations are left to the reader; and it seems doubtful to me that the exercise thus required from the reader will be as beneficial as a more ample presentation in the text would have been. The material of the second half requires considerable amplification, as noted, not only in providing details of the topics mentioned but in introducing other, unmentioned topics if a rounded view of geometry is the goal.

In view of the difficulty that the human race as a whole has experienced in writing introductions to Riemannian geometry, it would seem churlish to harp further upon any perceived limitations of a particular instance. This book is an attractively and cleanly written addition to the surprisingly small literature of geometry books at this level. A suitably prepared reader will find it a straightforward path to topics of central concern in Riemannian geometry. (There are a large number of solecisms arising from what appears to be direct translation from French. The authors are blameless in this, in my view, but Springer should be ashamed not to have done a decent job of copyediting.) In the face of the importance of the subject matter, it has been surprising how difficult it has continued to be to make one's initial foray into Riemannian geometry. While there may be no royal road to geometry, this book offers at least some clear signposts to readers. But its terseness leaves them to walk a great many lonesome valleys by themselves.

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The complex analytic theory of Teichmüller spaces, by Subhashis Nag. John Wiley, New York, Chichester, Brisbane, Toronto, Singapore (Canadian Mathematical Society Series of Monographs and Advanced Texts), 1988, xii + 427 pp., $54.95. ISBN 0-471-62773-9

Teichmüller theory often reminds me of a mathematical foundling. It first appeared, albeit in a different guise, in the attempt by Friedrich Schottky to prove that Riemann surfaces (qua algebraic curves) can be uniformized (parametrized) by meromorphic functions defined in plane domains. According to Felix Klein, Weierstrass, as Schottky's thesis advisor, rejected this (quite correct) argument and removed it from the thesis. So Teichmüller theory was almost stillborn. It explicitly appears first in the work of Klein's collaborator, Robert Fricke. The theory was discovered anew by Oswald Teichmüller around 1940 and reached maturity around 1960 under the tender care of Lars Ahlfors, Lipman Bers and their students following pioneering work of Ernie Rauch.

Almost immediately the theory found a series of foster parents. First, algebraic geometers took us, the noble but isolated practitioners of this iconoclastic discipline, under their mightly wings. We learned the joys of providing lemmas solving partial differential and integral equations and