viewpoints or aspects cf. P. T. Johnstone [**10**] and J. R. Isbell [**8**]), I believe that topologists will remain in doubt whether it is worth the effort to study all categorical notions provided in this book, and that categorists will say that indeed too small a portion of their subject has been presented, even if repetitiously. As a researcher I welcome Preuss' book as a reference manual on the great many notions on generalized topological structures used in the literature, but I am disappointed that the opportunity to provide genuine guidance through a new field, in which the important material still needs to be selected from the many concepts offered, has not been used to its fullest.

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Multidimensional Brownian excursions and potential theory, by K. Burdzy. Longman Scientific & Technical, Essex, New York, (Pitman Research Notes in Mathematics Series, vol. 164), 1987, 172 pp., $44.95. ISBN 0-582-00573-6

Burdzy's monograph deals with a recent addition to the probabilistic arsenal, Brownian excursion laws, and their application to boundary problems in classical potential theory and complex analysis. Excursion laws first arose in the work of K. Itô, and later in that of B. Maisonneuve.

Itô and Maisonneuve developed a method for analyzing certain stochastic processes, by breaking them up into smaller, more tractable pieces. The pieces, properly called excursions, are characterized by certain σ-finite measures. Burdzy abstracts the properties of these measures, to produce a larger family of measures which he calls excursion laws. He studies these objects systematically in their own right and applies them to certain problems from potential theory. Instead of starting with the abstraction, we'll first discuss the basic connections between Brownian motion and analysis, and then summarize the work of Itô and Maisonneuve.

Brownian motion. A stochastic process is essentially the trajectory of a particle moving at random in some state space. We'll follow common usage, and refer to properties of the process even when we really mean properties of its law (that is, of the measure that assigns numbers (probabilities) to sets of possible trajectories). Assume that the future evolution of the process depends on the past only through the present state of the process. More precisely, we assume the following (the *strong Markov property*): For a large class of random times T (called *stopping times*; any deterministic time t is one, as are times such as the first time the process encounters a fixed set), our best prediction of the future trajectory $(Y(t) = X(T + t), t \geq 0)$, even given knowledge of the entire past $(X(s), s \leq T)$, depends only on the present state $X(T)$. Moreover, probabilities involving this prediction are those the original process would give, were it started off in state $X(T)$ at time 0. We call X a *Markov process*.

Among Markov processes moving continuously in $\mathbf{R}^n$, one is singled out as canonical: *Brownian motion* is, up to a normalization transformation, the unique such process whose random fluctuations are homogeneous and isotropic in space.

Kakutani realized that this spatial homogeneity forges a link with analysis. A function $h(x)$ is *harmonic* if its average value over any sphere equals its value at the sphere's centre. Harmonic functions arise in fields from PDE's and electrostatics, to functions of a complex variable. In probabilistic terms, $f(x)$ is the average value of f at the first place a Brownian motion encounters ∂B, where the motion starts at x and B is any sphere centred at x. So much is definition, yet the property holds as well for arbitrary domains D containing x. This gives a probabilistic construction of the solution to the Dirichlet problem, that is, of a harmonic function with prescribed values on the boundary of a given domain. Adding a layer of structure often lets one understand old concepts more clearly. In our case, we find simple interpretations of the classical notions of harmonic measure, and regular boundary points; The former is simply the law of the first place the motion hits ∂D (this law depends on the initial point of the motion, so we really obtain a family of equivalent measures). The latter are points, starting from which the motion hits ∂D immediately. Similarly, the electrostatic *capacity* of a set (which determines the charge that can be stored in a capacitor built in the shape of that set) acquires an interpretation as the probability that a Brownian motion will hit the set, starting "from infinity."

Brownian motion can be used to study the boundary values of functions h which are positive and harmonic on a domain D. If D is the unit disk, then Fatou's theorem states that h has a limit along any curve that approaches a boundary point of D in a nontangential manner. What Burdzy calls a *minimal-fine limit* gives a replacement for nontangential approaches, less tied to geometry. Positive harmonic functions still have minimal-fine limits at almost every boundary point, and this remains true if the unit disk is replaced by an arbitrary domain. Now the geometric Euclidean boundary is inadequate; it must be replaced by the potential theoretic *Martin boundary*. For Lipschitz domains, the two coincide. Now "almost every" refers to harmonic measure.

To go further, one needs to tie down the location at which the Brownian motion first leaves D. Doob accomplished this, introducing the notion of an *h-transform*. First we modify the Brownian motion, killing it upon leaving D. Upon first doing so, the particle evaporates, instantly jumping to some added "cemetery" state, from which it never returns. Now let h be positive and harmonic. An h-transform is a new stochastic process, whose trajectories resemble those of killed Brownian motion, except that for small balls B centred at $x \in D$, the law of the first exit from B starting at x is $h(x)^{-1}h(y)\sigma(dy)$, σ being normalized surface area on ∂B. Thus the h-transform is driven towards places where h is large. If $A \subset \partial D$ has positive harmonic measure, let h have boundary value 1 on A, and 0 elsewhere. Then an h-transform is forced to leave D through A. There is another way of producing a process with this property: use *conditional probability*, and "condition" the killed Brownian motion to leave D through A. The two turn out to be the same.

A function which is positive and harmonic in D is *minimal* if it is not a sum of two other such functions (unless of course the summands are multiples of the original function). When D is the unit disk, these are given by the *Poisson kernel*: $h_y(x) = (1 - |x|^2)|x - y|^{-n}$. Here $y \in \partial D$ is the *pole* of h_y. In general there is a one-to-one correspondence between minimal positive harmonic functions h and their poles y, the latter now being points of the (minimal) Martin boundary. Choquet's theorem then gives an integral representation of positive harmonic functions, in terms of measures on this boundary. As we expect, if h is minimal with pole y, then an h-transform will exit D at y. Despite the fact that $A = \{y\}$ has harmonic measure zero, so is almost surely missed by killed Brownian motion, we still think of an h-transform as a Brownian motion "conditioned to leave D at y." The Martin boundary then acquires a probabilistic interpretation as a parametrization of the different ways Brownian motion can be conditioned to leave D. The potential-theoretic notion of a function f having a minimal-fine limit l at y turns out to be equivalent to $f(X(t))$ approaching l almost surely, as $X(t)$ approaches ∂D, where X is a transform by the minimal harmonic function with pole at y. Similarly, the key technical concept used by Burdzy, that of a set U being *minimal-thin* at y, can be taken to mean that for some initial point x, X has positive probability of missing U on its journey out to ∂D. See Doob(**1984**) for details. Durrett(**1984**) provides an excellent introduction to Brownian motion and its applications in analysis.

Excursion theory. Suppose now that $(X(t))$ is merely a Markov process, and that we have singled out a state a. We call the trajectory of X, between successive visits to a, an *excursion* of X away from a. If, with probability one, X visits a repeatedly but only at a discrete set of times, then we have a first, second, third, ... excursion and the Markov property implies that they are all independent of each other. In this case, there are standard methods for extracting information about the process from the common law H of these excursions. We think of H as the law of a "typical" excursion, but note also that if Λ is a set of possible trajectories, then $kH(\Lambda)$ is the average number of excursions, among the first k, that follow trajectories from Λ.

Unfortunately, this does us little good when applied to Brownian motion in $\mathbf{R}$. Regardless of a, the set of times t such that $X(t) = a$ almost surely has the topological structure of a Cantor set; each excursion has another arbitrarily close to it. Thus we have no way of labelling the excursions as "first," "second," and cannot single out a "typical" excursion. For example, though we might choose to examine the excursion underway at some fixed time t, this selection predisposes the excursion to be unusually long.

Itô(**1971**) circumvented this problem by using *local time* at a. For Brownian motion, $\{s; s \leq t, X(s) = a\}$ has Lebesgue measure zero for each fixed t, yet there is a deterministic function g such that if this set is nonempty then its g-Hausdorff measure (call it $L(t)$) is almost surely finite and nonzero. $L(t)$ gives a local time at a, that is, a way for a clock to keep time if it can only tick when X is at a. The excursions turn out to come homogeneously and independently when we keep time with this clock.

We can no longer define the law of a "typical" excursion. Instead we work with the expected number of excursions that follow a trajectory from Λ, and come during a fixed interval of local time. Itô showed that the number of such excursions coming in disjoint intervals of local time will be independent of each other, and that the expected number coming in an interval $[s, t]$ will be $(t - s)H(\Lambda)$, for some measure H. Since infinitely many excursions may come in every open interval of local time, H may be an infinite measure (it is for one dimensional Brownian motion). Nonetheless, it has many of the same uses as our earlier H. For example, the first excursion of duration longer than δ has a law that can be expressed using H. It will be H restricted to the set Λ_δ of excursions lasting at least time δ, but multiplied by a constant that gets small as $\delta \downarrow 0$. This factor arises because $H(\Lambda_\delta)$ is the rate at which such excursions happen (in the local time scale), and as $\delta \downarrow 0$, they come increasingly rapidly. For a taste of what can be done using H, see Williams(**1979**).

Maisonneuve(**1975**) generalized Itô's decomposition, treating excursions away from more general sets M than $\{a\}$. Again, an excursion is the trajectory followed by the process during a maximal interval during which the process doesn't visit M. For Brownian motion in $\mathbf{R}^n$, and M the complement of a domain D, one can define a local time on M, and use it as a clock with which to time the starts of excursions into D. The excursions will no longer have the independence property described above, as they are tied together by the initial and final positions of the Brownian motion during the excursion. Nonetheless, Maisonneuve gave a formula that allows one to compute with them. The role of H in Itô's setup is now played by a family of σ-finite measures H^x, one for each possible initial point $x \in M$ of an excursion. The H^x have proved useful in understanding diffusions that reflect, or bounce off the "barrier" M.

Excursion laws. The H and H^x are all examples of excursion laws on D, as defined in Burdzy's book. That is, they are σ-finite measures having the strong Markov property at strictly positive stopping times, with respect to the transition mechanism of killed Brownian motion. Formally, let Ω consist of all possible excursion trajectories. That is, it is the set of $\omega :]0, \infty[\to D \cup \Delta$ such that ω moves continuously in D until such time (if any) as it jumps to the cemetery Δ, where it stays ever after. Brownian motion, started at x and killed upon leaving D, has a law P^x which we can realize as a probability measure on Ω. A σ-finite measure H on Ω is called an *excursion law* if for every stopping time $T(\omega) > 0$ (T may be random, so is a function of the trajectory ω), and every suitably measurable $A, B \subset \Omega$,

$$H\{\omega; \omega(\cdot \wedge T(\omega)) \in A, \omega(\cdot + T(\omega)) \in B\}$$
$$= \int_{\{\omega; \omega(\cdot \wedge T(\omega)) \in A\}} P^{\omega(T(\omega))}(B) H(d\omega).$$

In the case of interest to us, H-almost all trajectories have the same limit x in the minimal Martin boundary of D, as $t \downarrow 0$. If this is so, we write H^x for H. The H^x are the objects of study in Burdzy's book.

Though others have used excursion laws to characterize excursions, Burdzy and his collaborators are the first to study the laws in and of themselves. Maisonneuve's construction essentially gave a nontrivial H^x for almost all $x \in \partial D$. Burdzy provides an intrinsic construction for *every* (minimal, accessible) x in the Martin boundary. Such an H^x is unique, up to a constant multiple, if it is *standard*: that is, if it hits any fixed compact subset of D with only finite probability. There is scope for pathology though. Many nonstandard H^x can exist, and in odd domains there are quite natural sets that can receive infinite mass from even standard excursion laws.

Minimal-thinness can be reformulated in the new language; U is minimal-thin at x iff the standard H^x gives zero mass to the set of trajectories that hit U instantaneously. Burdzy gives a host of concrete formulae in the case of D the half-space. He can apply some of these more generally, as he shows that H^x for a C^2 domain shares the same "local properties" as H^x for a half-space. For a C^1 domain this may fail for almost every x. In some cases, H^x can be constructed as a rescaled limit of the P^z, as z approaches x from inside D. The central result of the book is an integral test for minimal-thinness in Lipschitz domains, ultimately based on Wiener's test. The illustrative application is to the "angular derivative problem," which asks when a conformal equivalence of complex domains has a derivative at a given boundary point. Carathéodory's "prime ends" appear as well.

Can one obtain the same applications, substituting h-transforms for Burdzy's excursion laws? Probably, especially since he uses h-transforms to construct the H^x in the first place. There is at least one reason not to do so though. His favourite technique is the "splicing together" of excursion laws in domains D_1, D_2 with $D_1 \triangle D_2$ small. This works well, as the transition mechanism is Brownian in both D_1 and D_2. A similar construction with h-transforms would be awkward and more technical. It is interesting to note that another σ-finite measure, due to Kuznetsov has recently provided analogous simplifications in the sometimes technical study of time-reversal. Perhaps excursion laws will prove to be as useful.

The book itself is nicely written and well organised. The existing literature consists of scattered articles by Burdzy and his colleagues, with various stages of evolution, both of hypotheses and results. The book sorts it all out very clearly and carefully. It provides little general background, instead referring to some of the sources we've listed. The book is photoreproduced, with no index, but the TEX is very readable.

In summary, excursion laws are interesting in themselves, yet arise naturally and have an elegant theory. As Burdzy's excellent treatise demonstrates, they may prove to be a valuable tool in analysis as well.

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Riemannian geometry, by S. Gallot, D. Hulin and J. Lafontaine. Universitext, Springer-Verlag, Berlin, Heidelberg, New York, 1987, xi + 248 pp., $29.00. ISBN 3-540-17923-2

"Of making many books there is no end." The intervening thousands of years have surely added the verification of experience to the Divine Authority behind that statement, and not least in mathematics. But for all the truly endless making of mathematics books, there remain gaps in the mathematical literature; there are still books one would like to see written. Many of these desiderata would be works at the intermediate graduate level. There is no shortage of elementary works, and whatever