6. J. Smoller, *Shock waves and reaction-diffusion equations*, Grundlehren Math. Wiss., vol. 258, Springer-Verlag, Berlin and New York, 1983.

7. T. Wazewsky, *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles*, Ann. Soc. Math. Polon. **20** (1947), 279–313.

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Quadratic and Hermitian forms, by Winfried Scharlau. Grundlehren der Mathematischen Wissenschaften, vol. 270, Springer-Verlag, Berlin, Heidelberg, New York and Tokyo, 1985, x + 421 pp., $84.00. ISBN 3-540-13724-6

When my own book [L] on quadratic forms appeared in 1973 (W. A. Benjamin, Inc.), A. Pfister's review of it in *Zentralblatt für Mathematik* began with a polite exclamation: "Ein neues Buch über quadratische Formen!" Having thus acknowledged my source, it nevertheless seemed appropriate for me to begin this review of Scharlau's book by exclaiming: "A new book on quadratic forms!"

The point is, of course, that a new book in quadratic forms is not like a new romance novel on the supermarket stand. Generally, one does not write a new book in mathematics unless one has something new to say, or at the very minimum, has something to say in a new way. In the arena of quadratic form theory, fortuitously, all authors, past and present, have adhered scrupulously to this principle. Thus, when a new book in quadratic forms appears, readers in the field greet the event with interest and considerable expectations.

So far, about a couple dozen books in quadratic forms have been written. Scharlau listed them chronologically in a special section in his bibliography. Using the year 1967 as a watershed, the list enumerates exactly 12 books written on or before 1967, and another 12 books written thereafter. This carefuly compiled list of books provides us an excellent vista point from which to view the historical development of the subject. In particular, before talking about Scharlau's book, it would be worthwhile to first take a look at this book list, to see what has been written on the subject before.

Looking through this list, one sees that few of the books among the two dozen duplicated others. Each book seemed to have its own focus in the vast subject of quadratic forms, from the days when the subject was a subdiscipline of number theory, to the modern age when the number-theoretic approach and the algebraic approach thrive together. In the pre-1967 list, the classical treatises of Lipschitz and Bachmann are probably rarely used by modern readers who (I can't blame them) prefer to deal with more-up-to-date terminology. Artin's *Geometric Algebra* and Dieudonné's *La Géométrie des Groupes Classiques* are no doubt great books by any standard, but one might argue if they can properly be classified as books on quadratic forms. Scharlau probably listed them as examples of books in algebra in which quadratic forms play a role behind the scenes. Bourbaki's *Algèbre, Chapitre* 9 contains a wealth of information on both sesquilinear and Hermitian forms, all presented in an uncompromisingly general setting. Chevalley's book on spinors is a uniquely original work; again it is not the easiest book to read, but if you want to find out about more exotic topics such as spinor representation, or the Principle of Triality, there is no better source. Then we have a very impressive list of books on the arithmetic-analytic aspects of quadratic form theory: Eichler, Jones, Watson, Siegel (Bombay Lectures), O'Meara, Cassels. These authors' treatises cover the triumphs of integral quadratic form theory from the classical through modern times. O'Meara's text, especially, succeeded spectacularly in unifying the subject and standardizing its methodology, and has thus influenced generations of students in quadratic forms and number theory. In the rest of the long list, one finds Bass' Tata Lecture Notes and Bak's book in the Princeton series: these cover aspects of quadratic form theory from a K-theoretic perspective. Milnor-Husemoller is another unique book in which the arithmetic, analytic and algebraic viewpoints are skillfully interwoven, to produce a treatment of quadratic form theory suitable, for instance, for topoplogical applications. Another work motivated by topology (but somehow omitted from Scharlau's list) is the lecture notes volume of Hirzebruch [H] devoted to the study of the connections between differentiable manifolds and integral quadratic forms. Then there is the series of books on the more modern algebraic theory of quadratic forms: Scharlau (Kingston notes 1969), Lorenz, Baeza, Marshall (Kingston notes 1980), Knebusch-Scharlau, Knebusch-Kolster, and two books by this reviewer. The coverage of these books range from quadratic form theory over fields, semilocal rings, to the abstract theory of forms and generic methods.

Now that we have quickly run through the book list, we can try to see where Scharlau's book fits in, and what are the new features it has to offer. Since the methods used in this book are predominantly algebraic (rather than arithmetic), I believe it should be classified as a contribution to the algebraic theory of quadratic forms. Scharlau offers the interesting viewpoint that the birth of the algebraic theory has to do with "the ideas of abstract algebra and abstract linear algebra introduced by Dedekind, Frobenius, E. Noether and Artin (which) led to today's structural mathematics with its emphasis on classification problems and general structural theorems." By all accounts, the modern algebraic theory of quadratic forms originated with Witt's 1937 paper in the Crelle Journal [W]. Witt was among the last of Emmy Noether's students in Göttingen, and wrote his dissertation under her in 1933. However, [W] seemed to have little to do with his dissertation on zeta functions of hypercomplex systems. Two important points of departure from classical treatments of quadratic forms were to make Witt's paper a revolutionary work. First, Witt's new theory was developed over an *arbitrary* ground field (of characteristic not 2) rather than some specific field like the rationals or a finite field; secondly, the main emphasis shifted from dealing with quadratic forms one at a time to dealing simultaneously with *all* quadratic forms (over some fixed field). Both of these themes, as one can see, bear the hallmarks of the general mathematical philosophy of the Noether-Artin school of abstract algebra. In the modern theory, of course, it also became important to *vary* the ground field and to study how the behavior of forms changes from one field to another. This "functorial" viewpoint was to emerge a few years after Witt's paper, in the seminal work of Eilenberg and Steenrod in algebraic topology.

The principal new object which emerges from Witt's study is what we now call the Witt ring, $W(F)$, of a field F, whose elements are basically the nonsingular quadratic forms over F, sorted out by a notion of similarity. (Two forms are Witt similar if and only if they have isometric anisotropic parts.) Witt proved a number of important theorems on the decomposition, cancellation and chain-equivalence of quadratic forms; however, he gave few examples of $W(F)$, and did not have the tools to analyze more deeply the structure of $W(F)$. Indeed, the next 30 years saw relatively little further progress, and the theory founded by Witt in 1937 remained essentially in its infant stage up to the first half of the 1960s. The next breakthrough came with Pfister's amazing solution of the "van der Waerden Problem" for nonreal fields. Inspired by Cassels' proof that $1 + x_1^2 + \cdots + x_n^2$ cannot be written as a sum of squares of n rational functions in $\mathbb{R}(x_1, \dots, x_n)$, Pfister succeeded in showing that all nonreal fields have level a power of 2, and conversely that any power of 2 is possible. The crux of his proof is the fact that, in any field F, the nonzero values of the quadratic form $\langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle$ always form a multiplicative subgroup of $\dot{F}$. The introduction of the basic forms $\langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle$ (now called n-fold Pfister forms) ushered in the modern age of the so-called algebraic theory. Using these forms as his main tool, Pfister developed a full-fledged structure theory of the Witt ring in his Habilitation thesis (1966), and proved the first local-global principle for quadratic forms over arbitrary fields **[P]**. Thus, after lying dormant for 30 years, the theory founded by Witt rose triumphantly to claim its rightful place in mathematics.

As the saying goes, the rest is history. The next two decades witnessed the tremendous growth of the algebraic theory in all directions. The theory of quadratic forms became a principal tool in studying the arithmetic of fields, and forged a strong union with Krull's general valuation theory on the one hand, and Artin-Schreier's theory of formally real fields on the other. New connections discovered between quadratic (and Hermitian) form theory with algebraic K-theory, surgery theory and L-groups, real algebraic geometry, model theory, Galois theory, Galois cohomology and the theory of finite-dimensional central simple algebras continued to expand the boundaries of the subject. Today the theory of quadratic forms is a vibrant research area which attracts many workers from both algebra and topology. Scharlau's graph on p. 417 of his book showing the dramatic rise in the number of journal pages devoted to this area of research since the mid-60s is perhaps the most vivid proof of the increasing intensity of the activity in this area.

Scharlau's new book provides a masterful introduction to the theory of quadratic forms with a modern outlook, covering many of the important research results discovered in the last two decades. In many instances, central results were extracted from full-length journal articles, and are presented to the reader in a clear and concise way. The basic algebraic theory is covered in the first four chapters. The topics presented include: structure of the Witt ring, formally real and ordered fields, Pfister's Local-Global Principle, the transfer of quadratic forms, generic methods and the theory of Pfister forms, field invariants, etc. The 160 or so pages occupied by these four chapters offer a very nice sampling of the most central results in the algebraic theory of quadratic forms over fields, and should be easily accessible to a beginning reader.

Though the main emphasis of the book is obviously on algebraic methods, the material in the text is by no means limited to the algebraic theory alone. Chapter 5 on rational quadratic forms and Chapter 6 on forms over Dedekind rings and global fields present classical delights such as Gauss sums, Hilbert reciprocity, Hasse-Minkowski Theorem, etc., which serve well to remind the reader of the number-theoretic origins of the subject. The value of the book is further enhanced by its two chapters (Chapter 7, Chapter 10) on Hermitian forms, presenting both the abstract categorical treatment and the more concrete Landherr classification theory for Hermitian forms on division algebras over global fields, both of which appear here in book form for the first time. Another chapter (Chapter 8) presents an efficient and largely self-contained treatment of the classical theory of finite dimensional central simple algebras, with emphasis on the existence, extension, and classification of involutions. Much of this material is a prerequisite to the Hermitian theory, since the classification of Hermitian forms over division algebras is so closely related to the classification of involutions on simple algebras. The integrated treatment of involutorial algebras and Hermitian forms as contained in Chapters 7, 8 and 10 is highly successful, and represents perhaps one of the most outstanding features of the book.

As usual, I suppose the reviewer's job is not complete without trying to pick on something. But in all fairness I can come up with only very minor criticisms. As a book in its first edition, this one certainly has its share of misprints (fortunately mostly of a nonmathematical nature). Occasionally, facts are used in earlier sections before they are proved in later sections or chapters. A couple of times, the author seemed to pull things out of a hat; for instance, the Witt invariant defined on p. 81 (without the benefit of Clifford algebras) by four tedious and unmotivated formulas seemed a little hard for an average reader to swallow. Lastly, occasional notational inconsistencies do exist in the text. For instance, the space of orderings of a field K is denoted sometimes by $\Omega(K)$ (p. 57) and other times by X_K (p. 124); the cone P of an ordering sometimes contains 0 (p. 107) and other times doesn't (p. 40). These inconsistences could easily have been avoided. The use of $B(K)$ (p. 86) for the subgroup of elements of order ≤ 2 in the Brauer group of a field K seems highly nonstandard; a more recognizable notation would have been ${}_2Br(K)$. But truthfully, none of these really detracts from the mathematical merits of the book. In particular, all is forgiven when you discover in §1 of Chapter 8 what is perhaps the most intriguing remark in the whole book, which reads in bold face "**1.7. No Remark**"!

Summing up, Scharlau has presented to the mathematical community a new book in quadratic forms which is well written, eminently readable and of excellent reference value. While obviously many important topics have to be left out, the selection and organization of the material overall were done thoughtfully and with good vision. The end-product is a book which succeeded for the first time in encompassing the theories of quadratic and Hermitian forms, with a very substantial and engaging coverage of both. A book like this usually has a very positive effect on the growth of a field. Obviously, researchers and students alike in the area of quadratic forms should thank Scharlau for his great effort.

REFERENCES

[H] F. Hirzebruch, W. D. Neumann and S. S. Koh, *Differentiable Manifolds and Quadratic Forms*, Marcel Dekker, New York, 1971.

[L] T. Y. Lam, *The Algebraic Theory of Quadratic Forms*, W. A. Benjamin, Inc., Reading, Mass., 1973. (Revised Printing: 1980.)

[P] A. Pfister, *Quadratische Formen in beliebigen Körpern*, Invent. Math. **1** (1966), 116–132.

[W] E. Witt, *Theorie der quadratischen Formen in beliebigen Körpern*, J. Reine Angew. Math. **176** (1937), 31–44.

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Collected mathematical papers of Jacob Nielson, edited by Vagn Lundsgaard Hansen. Birkhäuser, Boston, Basel, Stuttgart, 1986, vol. 1, 1913–1932, xix + 458 pp., sFr. 198. ISBN 0-8176-3140-2, vol. 2, 1932–1955, xi + 440 pp., sFr. 192. ISBN 0-8176-3151-8

Introduction. This valuable two volume set *Jakob Nielsen: Collected mathematical papers* contains thirty seven articles originally published between 1913 and 1955. Several of the papers appear for the first time in translation from the original German and Danish. The volumes also contain a brief biographical sketch and five essays on the impact of Nielsen's work.

The five essays review Nielsen's work in four different fields. The essay by Werner Fenchel outlines Nielsen's contributions to the theory of discontinuous groups of isometries of the hyperbolic plane and the geometry of the surfaces determined by such groups. Bruce Chandler and Wilhelm Magnus outline the purely group theoretic results of Nielsen and describe