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The homotopy index and partial differential equations, by Krzysztof P. Rybakowski. Universitext, Springer-Verlag, Berlin, Heidelberg, New York, 1987, xii + 208 pp., $39.50. ISBN 3-540-18067-2

"Toutes les voies diverses où je m'étais engagé successivement me conduisaient à l'Analysis situs. J'avais besoin des données de cette science pour poursuivre mes études sur les courbes définies par les équations différentielles et pour les étendre aux équations différentielles d'ordre supérieur et en particulier à celles du problème des trois corps." This is the way Poincaré describes, in 1901, the role of topology in the genesis of the qualitative theory of differential equations, and the fruitful connection between the two areas has been of increasing importance since.

The basic concepts of general topology have found their use in the pioneering contribution of Birkhoff on dynamical systems considered as continuous flows in metric spaces, providing a unified setting for the various types of flows defined by differential equations, Volterra equations or iterates of mappings. We owe to Birkhoff the important notions of limit sets, invariant sets and recurrent motions. Recall that a *dynamical system* or *flow* on a metric space X is a continuous mapping $\pi: X \times \mathbf{R} \to X$ such that $\pi(\cdot, 0) = I_X$ and $\pi(\pi(x,t),s) = \pi(x, t+s)$ for all $x \in X$ and $t, s \in \mathbf{R}$. If, for each $x \in \mathbf{R}^n$, the Cauchy problem

$$y' = f(y), \qquad y(0) = x, \tag{1}$$

has a unique solution $y(t;x)$ defined for all $t \in \mathbf{R}$, then the mapping $\pi: (x,t) \mapsto y(t;x)$ is a dynamical system on $\mathbf{R}^n$. A subset $A \subset X$ is said to be an *invariant set* for π if $\pi(t,x) \in A$ when $t \in \mathbf{R}$ and $x \in A$. Important examples of invariant sets for the dynamical system associated to (1) are its equilibria (i.e. the zeros of f) and its periodic orbits.

The relation of algebraic topology with the study of nonlinear differential equations is not less important and, already in 1883, Poincaré made explicit use of Kronecker's characteristic, the forerunner of Brouwer's topological degree, to prove the existence of periodic solutions in the three-body problem. This degree can also be used to study the zeros of a continuous vector field f on $\mathbf{R}^n$, i.e., the equilibria of the associated differential equations. Namely, given a bounded open set B with no zeros of f on its boundary, the Brouwer degree of f in B is an algebraic count of the number of zeros of f in B, and in particular is equal to zero when f does not vanish in B; it only depends on the behavior of f on the boundary ∂B of B and is insensitive to small perturbations of f (and hence to continuous deformations of f which do not introduce zeros on ∂B) (see e.g. **[4]** for precise definitions and extensions). When f is the gradient of some real function F on $\mathbf{R}^n$, the *Morse index* (see e.g. **[3]**) provides a sharper tool to analyze the zeros of f, i.e. the critical points of F, when they satisfy a nondegeneracy condition.

Another situation was considered in the late forties by Wazewsky [7], for the ordinary differential equation

$$(2) \qquad y' = f(y)$$

with f locally Lipschitzian on $\mathbf{R}^n$. Taking for simplicity $n = 2$, assume that we can find a rectangle B such that the field f points outside B on two opposite sides of ∂B, inside B on the two other sides, the corner points being bounce-off points. Let us call B^- the set of exit and bounce-off points on ∂B and let C be a continuous curve in B joining points of B^- belonging to different sides of ∂B. Then C contains a point x_0 such that the corresponding solution $y(t; x_0)$ of (2) must stay in B for all $t \geq 0$. Indeed, if it is not the case, then, for each $x_0 \in C$, the exit point $g(x_0)$ of the corresponding solution belongs to B^-, and we define in this way a (continuous) mapping $g: C \to B^-$ for which the points of $C \cap B^-$ are fixed. Thus, $g(C)$ is not connected, a contradiction. Wazewsky's method uses information on the behavior of f on the boundary of B to provide information on the behavior of some solutions of (2) inside B, namely solutions which stay in B for all $t \geq 0$.

The special cases considered above motivated Conley and Easton in their pioneering work of 1971 [1]. Given a bounded invariant set $K \subset \mathbf{R}^n$ for the flow associated to (2), they call K an *isolated invariant set* if there exists a closed neighbourhood N of K such that K is the largest invariant set of (2) in N, which is then called itself an *isolating neighbourhood*. In particular, such a N is said to be an *isolating block* if, on ∂N, the solution immediately leaves N in one or the other time direction. An important result is that, given K and N as above, there will always exist an isolating block B such that $K \subset B \subset N$. For such a B, let us denote again by B^- the set of points of ∂B such that the solution leaves B for positive t. It can be shown that, for all the isolating blocks B of K, the pointed spaces $(B/B^-, [B^-])$ are homotopically equivalent, where B/B^- is obtained by collapsing B^- in B to one point, and $[B^-]$ is the collapsed set B^-. The homotopy type $[B/B^-, [B]]$ of $(B/B^-, [B^-])$ is then called the *homotopy* or *Conley index* $h(K)$ of K.

For example, if we consider the following linear system in R^2,

$$y' = \operatorname{diag}(\lambda_1, \lambda_2) y$$

with $\lambda_1 > 0 > \lambda_2$, then $K = \{(0,0)\}$ is an invariant set and, for any $r > 0$, $B = [-r, r] \times [-r, r]$ is an isolating block for K, with B^- made of two opposite sides of B. Hence, $h(\{(0,0)\}) = \Sigma^1$, if we denote by Σ^n the homotopy type of the pointed n-sphere. More generally, for the corresponding system in $\mathbf{R}^n$ with k positive eigenvalues and $n-k$ negative eigenvalues, one has $h(\{(0,0)\}) = \Sigma^k$. Now, if $\bar{0}$ denotes the homotopy type of the pointed element set $(\{p\}, p)$, a basic existence result holds for the Conley index, namely that $K \neq \emptyset$ if $h(K) \neq \bar{0}$. If we also notice that the Conley index enjoys a continuation property, we shall easily be convinced that, like the Brouwer degree for equilibrium solutions, the Conley index will be a flexible existence tool for bounded invariant sets.

The above notions and properties can and have been extended to the case of a flow on a compact metric space X and the corresponding homotopy index generalizes the classical Morse index on a compact manifold by assigning an index not only to nondegenerate equilibria of gradient flows, but also to arbitrary isolated invariant sets of arbitrary flows on compact spaces (see e.g. [**2** and **6**] for more details and references). Now, removing the compactness assumption on the underlying space in fixed point or critical point theories requires the introduction of some restriction upon the class of mappings which are considered. To extend the Brouwer degree to the case of mappings defined on a Banach space, Leray and Schauder had to deal with compact perturbations of identity. To extend the Lusternik-Schnirelmann or the Morse theory to functionals on a Banach manifold, Palais and Smale had to consider functionals verifying their condition C, now called the Palais-Smale condition. The situation is similar when one wants to extend the Conley index to local semiflows on not necessarily compact metric spaces, and it was first done by Rybakowski in the early eighties [**5**], by introducing a concept of *admissibility* with respect to a semiflow for closed subsets of a Banach space.

The present monograph is a detailed exposition of the corresponding generalized Conely or homotopy index for invariant sets having an admissible isolating block, and of its applications to nonlinear partial differential equations. For example, questions like that of the existence of nontrivial solutions of asymptotically linear elliptic equations are treated in detail, as well as the existence of periodic solutions of second-order gradient systems of ordinary differential equations when degree theory is not applicable. Other applications can be found in the author's papers listed in the bibliography. The reader will also find in the book a detailed comparison between the Morse and homotopy indices for gradient flows on Hilbert manifolds, and between the Conley index and the Brouwer degree when both are defined.

The style of this carefully written monograph is more analytical than geometrical, and therefore provides a presentation of the homotopy index which is complementary to the geometric one associated to Conley's school. Therefore, besides earlier references like [2 and 6], Rybakowski's book will be instrumental in making more mathematicians familiar with this promising technique in dynamical systems and global analysis.

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Quadratic and Hermitian forms, by Winfried Scharlau. Grundlehren der Mathematischen Wissenschaften, vol. 270, Springer-Verlag, Berlin, Heidelberg, New York and Tokyo, 1985, x + 421 pp., $84.00. ISBN 3-540-13724-6

When my own book [L] on quadratic forms appeared in 1973 (W. A. Benjamin, Inc.), A. Pfister's review of it in *Zentralblatt für Mathematik* began with a polite exclamation: "Ein neues Buch über quadratische Formen!" Having thus acknowledged my source, it nevertheless seemed appropriate for me to begin this review of Scharlau's book by exclaiming: "A new book on quadratic forms!"

The point is, of course, that a new book in quadratic forms is not like a new romance novel on the supermarket stand. Generally, one does not write a new book in mathematics unless one has something new to say, or at the very minimum, has something to say in a new way. In the arena of quadratic form theory, fortuitously, all authors, past and present, have adhered scrupulously to this principle. Thus, when a new book in quadratic forms appears, readers in the field greet the event with interest and considerable expectations.

So far, about a couple dozen books in quadratic forms have been written. Scharlau listed them chronologically in a special section in his bibliography. Using the year 1967 as a watershed, the list enumerates exactly 12 books written on or before 1967, and another 12 books written thereafter. This carefuly compiled list of books provides us an excellent vista point from which to view the historical development of the subject. In particular, before talking about Scharlau's book, it would be worthwhile to first take a look at this book list, to see what has been written on the subject before.

Looking through this list, one sees that few of the books among the two dozen duplicated others. Each book seemed to have its own focus in the vast subject of quadratic forms, from the days when the subject was a subdiscipline of number theory, to the modern age when the number-theoretic approach and the algebraic approach thrive together. In the pre-1967 list, the classical treatises of Lipschitz and Bachmann are probably rarely used by modern readers who (I can't blame them) prefer to deal with more-up-to-date terminology. Artin's *Geometric Algebra* and Dieudonné's *La Géométrie des Groupes Classiques* are no doubt great books by any