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AD AND THE VERY FINE STRUCTURE OF $L(\mathbb{R})$

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ABSTRACT. We announce results concerning the detailed analysis of $L(\mathbb{R})$ a considerable distance along its constructibility hierarchy, assuming the axiom of determinacy. Our main focus here is on the projective hierarchy, and specifically the sizes and properties of the projective ordinals. In particular, assuming determinacy, we calculate the values of the projective ordinals, getting

$$\boldsymbol{\delta}^1_{2n+1} = \aleph_{\left[\left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}\right\}2n-1+1\right]}.$$

1. Introduction. We let $\mathbb{R} = \omega^\omega =$: the set of infinite sequences of natural numbers, called "reals." We let $\boldsymbol{\Sigma}^0_1$ denote the collection of open subsets of $\mathbb{R}$, and $\boldsymbol{\Pi}^0_1$ the closed sets. We further let $\boldsymbol{\Sigma}^1_n$ denote the collection of continuous images of $\boldsymbol{\Pi}^1_{n-1}$ sets (where $\boldsymbol{\Pi}^1_0 \equiv \boldsymbol{\Pi}^0_1$) and $\boldsymbol{\Pi}^1_n$ the collection of complements of $\boldsymbol{\Sigma}^1_n$ sets. We say a set is *projective* if it is $\boldsymbol{\Sigma}^1_n$ for some n. Classically, with the work of Baire, Borel, Lebesgue, Lusin, Sierpiński, Suslin, and others, descriptive set theory emerged as the study of the structural properties of these sets. The hope here is that important set-theoretic problems which seem intractable or for which the answer seems pathological in general may become more amenable when restricted to the projective sets—these sets being explicitly definable. One such example which derives from the continuum problem is the perfect set problem, which asks whether every uncountable set contains a perfect subset (and hence has cardinality $2^{\aleph_0}$). Although the answer is easily seen to be negative assuming the axiom of choice (AC), the construction results in a pathological set, not explicitly definable.

In fact, one might hope that a reasonable theory would exist for a collection of sets extending considerably the projective sets. We let $L(\mathbb{R})$ denote the smallest inner model (i.e., transitive class containing the ordinals) of ZF set theory containing the reals, $\mathbb{R}$. Just as every set in Gödel's model L is definable from an ordinal, every set in $L(\mathbb{R})$ is definable from an ordinal and a real, and consequently one might expect a reasonable theory to exist for the sets of reals in $L(\mathbb{R})$. The projective sets, in fact, represent the first few levels in a natural definability hierarchy for the sets in $L(\mathbb{R})$ (see §3), and thus the study of the projective sets embeds into the larger problem of obtaining a structural theory for the model $L(\mathbb{R})$.

Unfortunately, working in ZFC (= ZF + AC; the set theory of "ordinary mathematics") there are strong limitations to the results obtainable. The

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early descriptive set theorists obtained a fairly complete theory of the first level of the projective hierarchy (i.e., $\mathbf{\Sigma}_1^1$ and $\mathbf{\Pi}_1^1$ sets) and settled some questions even at the second level. For example, they proved the perfect set theorem for $\mathbf{\Sigma}_1^1$ sets, and the measurability of $\mathbf{\Pi}_1^1, \mathbf{\Sigma}_1^1$ sets. However, many questions at the second level and essentially all at the third level are undecidable in ZFC, as can be shown using modern metamathematical techniques. Thus, to extend the classical theory further one needs to work with stronger axioms of set theory. One such axiom which concerns us here is the *axiom of determinacy*, or AD, which we define next.

To each $A \subseteq \omega^\omega$ we associate a two player game G_A:

I	$\alpha(0)$		$\alpha(2)$		$\ldots$
II		$\alpha(1)$		$\alpha(3)$	$\ldots$

$$\alpha = (\alpha(0), \alpha(1), \ldots)$$

Here, I and II alternately play natural numbers $\alpha(i)$, thereby ultimately producing a real α. We say I wins a run of the game G_A iff the resulting α is in A. The notion of a winning *strategy* is defined in the usual manner. AD asserts that for every $A \subseteq \omega^\omega$, G_A is *determined*, that is, one of the players has a winning strategy. This axiom was introduced in the 60s by Mycielski and Steinhaus, but first became an active subject of investigation in the late 60s through the work of Martin, Moschovakis, Solovay, and later Kechris, Steel and others. We refer the reader to [7] for an account of descriptive set theory, including AD and the structural theory of the projective sets which follows from it.

Although AD contradicts the axiom of choice, and thus fails in the universe of all sets, it has been proposed as the "correct" axiom for the submodel $L(\mathbb{R})$. Recent results of Martin, Steel, and Woodin in fact show that [ZFC + certain large cardinal axioms] implies "$\text{AD}^{L(\mathbb{R})}$," the assertion that all games in $L(\mathbb{R})$ are determined. This result lends further credibility to AD as a axiom for the model $L(\mathbb{R})$.

An extensive theory of $L(\mathbb{R})$-including a theory of the projective sets—has been developed on the basis of AD, buttressing the hope that AD will yield a complete and detailed theory of this model. This paper announces results which contribute to the development of this theory. We return first, however, to the projective case.

2. The projective hierarchy. Several important questions about projective sets were left unanswered by the theory previously developed from AD. To discuss these, we define a set of distinguished ordinals called the *projective ordinals*. We let $\mathbf{\Delta}_n^1 = \mathbf{\Sigma}_n^1 \cap \mathbf{\Pi}_n^1$ and define $\boldsymbol{\delta}_n^1$ = the supremum of the lengths of the $\mathbf{\Delta}_n^1$ prewellorderings of $\mathbb{R}$, where a prewellordering is a well-ordering on equivalence classes (in the absence of AC we must work with prewellorderings rather than well-orderings). We may motivate the projective ordinals by considering the continuum problem. In ZFC, $\Theta =:$ the supremum of the lengths of the prewellorderings of $\mathbb{R} = (2^{\aleph_0})^+$. The $\boldsymbol{\delta}_n^1$ thus describe a "definable" analog of the continuum problem as they invove restricting theprewellorderings to certain definable pointclasses. The

$\boldsymbol{\delta}^1_n$ are also important as they give structural information about the projective sets. For example, every $\boldsymbol{\Sigma}^1_{2n+1}$ set is $(\boldsymbol{\delta}^1_{2n+1})^+$-Borel (under AD), where we recall that a set is κ-Borel if it belongs to the smallest collection containing the open and closed sets, and closed under unions and intersections of length $< \kappa$.

The first projective ordinal may be computed in ZFC, the result being $\boldsymbol{\delta}^1_1 = \omega_1$. The exciting possibility was raised in the late 60s that, assuming AD, all the $\boldsymbol{\delta}^1_n$ could be computed explicitly in terms of the aleph (cardinal) function. The idea is to work in $L(\mathbb{R})$ with the axiom AD, and use the fact that the $\boldsymbol{\delta}^1_n$ are the same as computed in $L(\mathbb{R})$ or in V (= the universe of all sets). Assuming AD, Martin first computed $\boldsymbol{\delta}^1_2 = \omega_2$, $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$, and Martin-Kunen obtained $\boldsymbol{\delta}^1_4 = \omega_{\omega+2}$. It was also shown (Martin-Kunen) that $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$ and (Kechris) $\boldsymbol{\delta}^1_{2n+1} = (\lambda_{2n+1})^+$, the successor of some cardinal of cofinality ω. Kunen originated a program in the early 70s for computing the $\boldsymbol{\delta}^1_n$. The program stalled, however, and the problem of computing $\boldsymbol{\delta}^1_5$ became the first Victoria Delfino problem **[11]**. Using the ideas of this program, and building on earlier work of Martin and the author, we have now computed these ordinals. We have

THEOREM (ZF + AD + DC).

$$\boldsymbol{\delta}^1_{2n+1} = \aleph_{\left[\left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}\right\}{2n-1}+1\right]} \quad \textit{for all } n \geq 1$$

(*recall* $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$).

Here DC, the axiom of dependent choices, is a weak form of AC which asserts that every ill-founded binary relation has an infinite descending chain. By a result of Kechris **[4]**, in $L(\mathbb{R})$, AD $\Rightarrow$ DC. In fact, much more information about the cardinal structure below $\aleph_{\varepsilon_0} = \sup_n \boldsymbol{\delta}^1_n$ is obtained along the way. One such result is that between $\boldsymbol{\delta}^1_{2n+1}$ and $\boldsymbol{\delta}^1_{2n+3}$ there are precisely $2^{n+1} - 1$ regular cardinals (for $n = 1$ they are $\aleph_{\omega\cdot 2+1}$ and $\aleph_{\omega^\omega+1}$). We state explicitly a classical consequence of this. Recall that successor cardinals are regular under ZFC. Hence, under ZFC + $\mathrm{AD}^{L(\mathbb{R})}$ there are only finitely many cardinals between $\boldsymbol{\delta}^1_n$ and $\boldsymbol{\delta}^1_{n+1}$. We get

COROLLARY (ZFC + $\mathrm{AD}^{L(\mathbb{R})}$). $\boldsymbol{\delta}^1_n < \aleph_\omega$.

Combining this with earlier results, we get

COROLLARY (ZFC + $\mathrm{AD}^{L(\mathbb{R})}$). *Every projective set is the union of fewer than $\aleph_\omega$ Borel sets.*

The bulk of the analysis for the general projective case appears in **[1]**, however a second (forthcoming) paper is necessary to finish the analysis. Our methods are sketched in §4 below. The reader may also consult **[2]** where the complete proof for $\boldsymbol{\delta}^1_5$ is given, or **[3]** as an illustration of the methods used.

3. Beyond the projective hierarchy. Our methods extend considerably beyond the projective hierarcy. We consider again $L(\mathbb{R})$. As with L, $L(\mathbb{R})$ can be stratified into a hierarchy. In fact, if we define

$L_0(\mathbb{R}) = \mathbb{R} \cup \{\text{sets of finite rank}\}$,

$L_\lambda(\mathbb{R}) = \bigcup_{\alpha<\lambda} L_\alpha(\mathbb{R})$ for λ a limit ordinal,

$L_{\lambda+1}(\mathbb{R}) = \{x : x \text{ is definable over } L_\lambda(\mathbb{R}) \text{ using parameters from } L_\lambda(\mathbb{R})\}$,

then we have $L(\mathbb{R}) = \bigcup_{\alpha\in ON} L_\alpha(\mathbb{R})$. The sets of reals in $L(\mathbb{R})$ fall into a definability hierarchy, as they are all Σ_n definable over $L_\alpha(\mathbb{R})$ for some n, α. The projective sets form the first ω levels of the hierarchy, as they are the sets Σ_n-definable over $L_0(\mathbb{R})$. Steel **[6]**—has developed a "fine structure theory" for $L(\mathbb{R})$ assuming ZF + AD **[9]**. This suffices to answer certain questions about $L(\mathbb{R})$, for example, it gives a complete description of the scale property in $L(\mathbb{R})$ (see [7] for the definitions). Other problems, however, such as whether every regular cardinal is measurable (recall that κ is measurable if κ carries a κ-complete, nontrivial ultrafilter) seem to require a more detailed understanding of $L(\mathbb{R})$.

Our results provide such a detailed analysis for an initial segment of the $L_\alpha(\mathbb{R})$ hierarchy. Exactly how far this enables one to go is not clear, and is the subject of current investigation. However, the author has verified that the theory extends through the Kleene ordinal $\kappa = o({}^3E)$ (see **[4]** for the definition), and in fact, considerably beyond. This analysis is quite involved, however, and has not yet been written up. One consequence is the solution to a problem of Moschovakis, who conjectured in ZF + AD + DC that the Kleene ordinal should be the least inaccessible cardinal (this is the seventh Victoria Delfino problem). Steel **[10]** had shown that the Kleene ordinal was the least inaccessible Suslin cardinal. Our methods allow one to show that these cardinals coincide. Hence

THEOREM (ZF + AD + DC). *$\kappa = o({}^3E)$ is the least inaccessible cardinal.*

Certain other conjectures, such as every regular cardinal being measurable, can also be verified as far as the theory extends.

4. Outline of methods used. We consider the projective hierarchy, where the results have largely been written up **[1]**, and the ideas are most easily seen.

We proceed by induction on n, and introduce two inductive hypotheses which we call I_{2n+1} and K_{2n+3}; they assert, among other things, that

$$\boldsymbol{\delta}^1_{2n+1} = \aleph_{\left[\left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}\right\}2n-1^{+1}\right]}.$$

For $n = 0$, these hypotheses reduce to known consequences of determinacy, the latter being a theorem of Martin. The plan is to assume I_{2n+3}, K_{2n+3} and prove I_{2n+3}, K_{2n+5}. In **[1]**, the upper bound for $\boldsymbol{\delta}^1_{2n+5}$ is obtained. This argument is divided into three distinct parts.

In the first part we introduce families of canonical measures on the $\boldsymbol{\delta}^1_{2n+1}$ and $\boldsymbol{\lambda}^1_{2n+1}$ (= the predecessor of $\boldsymbol{\delta}^1_{2n+1}$) and prove two embedding theorems which—combined with a result of Kunen—reduce the computation of $\boldsymbol{\delta}^1_{2n+5}$ to the computation of the ultrapowers of $\boldsymbol{\delta}^1_{2n+3}$ by the canonical measures.

In the second part, we analyze functions $F : \boldsymbol{\delta}^1_{2n+3} \to \boldsymbol{\delta}^1_{2n+3}$ with respect to the canonical measures. To do this, we introduce a set $\mathscr{D}$ of finitary objects called *descriptions*. Each description gives rise canonically to a function $F : \boldsymbol{\delta}^1_{2n+3} \to \boldsymbol{\delta}^1_{2n+3}$ which it "describes." We then prove our *main theorem* which reduces the computation of the ultrapowers of $\boldsymbol{\delta}^1_{2n+3}$ by the canonical measures to the computation of the rank of a countable well-founded relation derived from $\mathscr{D}$.

In the third part, we compute the rank of this relation, the result being

$$\left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}\right\} 2n+3.$$

Via our main theorem, this gives the desired upper bound for $\boldsymbol{\delta}^1_{2n+5}$.

In a forthcoming paper, we will extend the analysis of functions in the second part above to analyze measures on $\boldsymbol{\delta}^1_{2n+3}$. This will allow us to complete the induction.

We recall a definition: we say $\kappa \to (\kappa)^\lambda$ if for all partitions $\mathscr{P}$ of the increasing functions from λ into κ into two pieces, there is a set of size κ homogeneous for $\mathscr{P}$. Along the way, we obtain

THEOREM (ZF + AD + DC). *Each $\boldsymbol{\delta}^1_{2n+1}$ has the strong partition property, $\boldsymbol{\delta}^1_{2n+1} \to (\boldsymbol{\delta}^1_{2n+3})^{\boldsymbol{\delta}^1_{2n+1}}$, and each $\boldsymbol{\delta}^1_{2n+2}$ has the weak property, $\boldsymbol{\delta}^1_{2n+2} \to (\boldsymbol{\delta}^1_{2n+2})^\lambda\ \forall\lambda < \boldsymbol{\delta}^1_{2n+2}$, but not the strong one.*

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