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GALOIS REPRESENTATIONS FOR HILBERT MODULAR FORMS

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Introduction. It is a basic problem of number theory to classify algebraic extensions of a number field F. For extensions with abelian Galois group, this is accomplished by class field theory. The main theorem of class field theory provides a canonical isomorphism between the Galois group $\mathrm{Gal}(F^{ab}/F)$, where F^{ab} is of the maximal extension of F with abelian Galois group, and the group $\pi_0(C_F)$ of generalized ideal classes. Although the nonabelian case of this theory remains largely undeveloped, the conjectures of Langlands provide a framework. A key step in this approach is to dualize, thereby viewing the isomorphism of class field theory as a correspondence between the (continuous) complex, one-dimensional representations of $\mathrm{Gal}(F^{ab}/F)$ and $\pi_0(C_F)$. Furthermore, L-series are attached to representations of both groups and Artin's reciprocity law asserts that these L-series coincide under the correspondence. The complex one-dimensional representations of $\pi_0(C_F)$ are of finite order and correspond to automorphic forms of a special type on GL_1, namely, to those whose infinity type is of finite order.

Let $\mathbf{A}_F$ be the adele ring of F. The considerations above lead to a more general problem: for all $n \geq 1$, to identify the L-functions of automorphic forms on $GL_n(\mathbf{A}_F)$ of *arithmetic type at infinity* (cf. **[BRn]**) with L-functions attached to n-dimensional *motivic* Galois representations. By a motivic representation we mean one which occurs as a subrepresentation of the (étale) cohomology of a smooth proper variety defined over F. All complex Galois representations with finite image are motivic, as are certain λ-adic representations with infinite image. Here we recall that a λ-adic representation is a continuous representation of $\mathrm{Gal}(\overline{F}/F)$ on a finite-dimensional vector space over a finite extension of $\mathbf{Q}_l$. Even for $n = 1$, this program goes beyond class field theory, because the one-dimensional motivic λ-adic representations may have infinite order (e.g., the representations provided by the Shimura-Taniyama theory of abelian varieties with complex multiplication). In this case, such a representation is Hodge-Tate **[F]** and hence, by a theorem of Tate, is locally algebraic. It is therefore associated to an algebraic Hecke character with the same L-function.

Observe that our problem consists of two parts. On the one hand, given a Galois representation ρ, one wants to construct an associated automorphic form $\pi(\rho)$. If $n = 2$ and ρ is a complex representation with solvable

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image, the existence of $\pi(\rho)$ is due to Langlands (as completed by Tunnell). However, not much is known beyond this. On the other hand, given an automorphic form π of arithmetic type, one seeks $\rho(\pi)$. For $n > 2$, little progress has been made. The case $n = 2$ has been much studied over the last three decades (see **[B]** for a survey of results). The purpose of this note is to give a motivic construction of $\rho(\pi)$ when $n = 2$, F is totally real, and π is of holomorphic discrete series type at infinity.

To describe our result, let π be a cuspidal automorphic representation of $GL_2(\mathbf{A}_F)$ such that for each infinite place v, π_v is a discrete series representation of weight k_v and central character $t \to t^{-w}$, where w is an integer independent of v (we normalize so that the lowest discrete series has weight 2). Then π corresponds to a holomorphic Hilbert modular newform of weight (k_v).

THEOREM 1. *Suppose the k_v and w are all congruent modulo 2. Then there exists a number field $T \subset \mathbf{C}$ and a collection $\rho(\pi) = \{\rho_\lambda\}$, where for each l-adic completion T_λ of T, ρ_λ is a continuous representation of* $\mathrm{Gal}(\overline{F}/F)$ *in $GL_2(T_\lambda)$ such that*

$$(*) \qquad L_v(s, \rho_\lambda) = L_v(s, \pi)$$

for all finite places v prime to l of F at which π_v is unramified. Furthermore, the system $\rho(\pi)$ is motivic, in the sense given above.

For $F = \mathbf{Q}$, Theorem 1 was obtained by Deligne **[D]** in 1969. For totally real number fields, the existence of $\rho(\pi)$ and its applications have been considered by several authors. The example of modular forms associated to Hecke characters of CM quadratic extensions of F suggests that the k_v must be congruent mod 2 if $\rho(\pi)$ exists. When $[F\colon \mathbf{Q}]$ is odd or π is discrete series at some finite place, the existence of $\rho(\pi)$, which is based on the work of Langlands and Shimura, has been known for several years **[O, RT]**. Recall that in these cases, $\rho(\pi)$ is realized in the étale cohomology of a fiber system of abelian varieties over a Shimura curve associated to a quaternion algebra B which is unramified at only one infinite place. This limits the method when $[F\colon \mathbf{Q}]$ is even, because B must then be ramified at an odd number of finite places.

A representation ρ_λ satisfying $(*)$ was constructed by Wiles **[W]** under the assumption that π_v is ordinary with respect to all v dividing l. R. Taylor **[T]** recently proved the existence of $\rho(\pi)$ without providing a motivic realization for it. In fact, Taylor shows, using work of Carayol **[C]**, that $(*)$ holds for all finite places v, i.e., $\rho(\pi)$ satisfies the Langlands correspondence everywhere locally. The works of Wiles and Taylor rely upon congruences between modular forms. In contrast, our construction gives a direct (but noncanonical) realization of $\rho(\pi)$ in the étale cohomology of a fiber system of abelian varieties defined over F. This provides additional information. For example, the ρ_λ are Hodge-Tate, by the work of Faltings **[F]**. One can in fact show that $\rho(\pi)$ is a Grothendieck motive over $\overline{\mathbf{Q}}$. Moreover, it will be applicable to self-dual cohomological cuspidal representations on $GL_n(\mathbf{A}_F)$ when the theory of the stable trace formula for unitary groups is developed for $n > 3$.

To explain our construction, let E/F be a quadratic CM extension and let $U(m)$ denote the quasi-split unitary group in m variables relative to E/F. Fix an infinite place w of F. By Landherr's theorem, there exists a unitary group $U^w(2n, 1)$ which has signature $(2n, 1)$ at w, is compact at the remaining infinite places, and is quasi-split at all finite places. The main point is the following: we can expect to find the l-adic representations associated to the L-function of the base change to $GL_{2n}(\mathbf{A}_E)$ of certain cuspidal representations of $U(2n)$ in the étale cohomology of local systems on a Shimura variety associated to $U^w(2n, 1)$. This would follow from the (conjectural) theories of endoscopy and the Hasse-Weil zeta function of the Shimura varieties attached to $U^w(2n, 1)$. We carry this out below for $n = 1$ by applying the theory recently developed in **[M]**. This theory uses work of Arthur, Kottwitz, Larsen, and Rapoport, as well as the results of **[R$_1$]**. Theorem 1 follows because of the close connection between GL_2 and $U(2)$.

1. Reduction to $U(1, 1)$. Let E be a quadratic extension of F and let π_E be the base change of π to $GL_2(\mathbf{A}_E)$.

1.1. Let X be a family of quadratic CM extensions E/F such that each finite place of F splits in at least one member of X. Suppose that for each $E/F \in X$, there exists a system $\rho(\pi_E) = \{\rho_\lambda\}$ of representations of $\mathrm{Gal}(\overline{F}/E)$ satisfying $(*)$ at (1) almost all places and (2) each finite place v prime to l of E of relative degree one over F such that $(\pi_E)_v$ is unramified. Then $\rho(\pi)$ as in Theorem 1 exists.

This follows from **[BRn]**, §4.2–4.3. If $\rho(\pi_E)$ is motivic for one E/F, then $\rho(\pi)$ is motivic. In fact, if $\rho(\pi_E)$ occurs $H^*(X)$ for a variety X over E, then $\mathrm{Ind}_F^E(\rho(\pi_E))$ occurs in $H^*(\mathrm{Res}_{E/F}(X))$. Since $\rho(\pi_E)$ is invariant under the conjugation for E/F, $\rho(\pi)$ occurs in $\mathrm{Ind}_F^E(\rho(\pi_E))$ and hence is motivic. We obtain below a family $\rho(\pi_E)$ for every E.

Henceforth, we assume, without loss of generality, that π_E is cuspidal.

1.2. Let $U = U(2)$ and let ψ denote the base change map for automorphic representations from $U(\mathbf{A}_F)$ to $GL_2(\mathbf{A}_E)$ **[R$_1$]**.
There exists a cuspidal representation π' of $U(\mathbf{A}_F)$ such that:

(a) for all infinite places v, π'_v is a discrete series representation.

(b) $\psi(\pi') = \pi_E \otimes \eta$ for some algebraic Hecke character η of E.

In fact, let μ be a Hecke character of E whose restriction to F is the character of order two $\omega_{E/F}$ associated to E/F by class field theory. Let $\varepsilon(\pi)$ be the representation $\pi^*(\sigma(g))$, where π^* is the contragredient of π and σ is the conjugation of E over F. By **[R$_1$]**, Theorem 11.4.1, if π is cuspidal on $GL_2(\mathbf{A}_E)$ and $\varepsilon(\pi) = \pi$, then either π or $\pi \otimes \mu$ lies in the image of ψ. Note that $\varepsilon(\pi_E \otimes \eta) = \pi_E \otimes \eta$ if $\eta \circ N_{E/F} = \chi(\pi)^{-1} \circ N_{E/F}$, where $\chi(\pi)$ is the central character of π. The existence of π' follows easily.

We show that if π' satisfies (a), then a compatible motivic system $\rho(\pi')$ exists such that $L_v(s, \rho(\pi')) = L_v(s, \psi(\pi'))$ for (1) almost all places and (2) each finite place v of E prime to l of relative degree one over F such that $\psi(\pi')_v$ is unramified. We may then take $\rho(\pi_E) = \rho(\pi') \otimes \eta^{-1}$ to prove Theorem 1 as in (1.1).

2. Endoscopy. Let $G = U(3)$ and let $H = U \times U(1)$. We denote by φ_H the endoscopic transfer map for automorphic L-packets from H to G. Extend π' to an automorphic representation π'' of H by projection on the first factor and let Π'' denote the L-packet on G corresponding to π'' via φ_H. Since π_E is cuspidal, it follows from [**R**$_1$], Theorem 13.3.2, that Π'' is cuspidal. Futhermore, for each infinite place v, $(\Pi'')_v$ is an L-packet of discrete series representations of G.

Let $G' = U^w(2,1)$. We now apply the analogue for the pair (G, G') of the Jacquet-Langlands correspondence: there exists an L-packet Π' on $G'(\mathbf{A}_F)$ such that $(\Pi')_v = (\Pi'')_v$ for all finite v and for $v = w$ [**R**$_1$, Corollary 14.4.2].

3. l-adic representations. Let G'' be the unitary similitude group associated to G'. Let $\Pi = \bigotimes \Pi_v$ be a cuspidal L-packet on G''. Let $\Pi_\infty = \bigotimes_{v|\infty} \Pi_v$ and assume that Π_w is of discrete series type with algebraic central character. Note that if $v|\infty$ and $v \neq w$, then Π_v consists of a single representation because G''_v is compact. The cardinality of Π_w and hence that of Π_∞ is 3. Let Π_f be the finite part of Π and let $\tau_f \in \Pi_f$. Let $d(\tau_f)$ be the number of elements $\tau_\infty \in \Pi_\infty$ such that $\tau_\infty \otimes \tau_f$ occurs in the space of cusp forms.

If the restriction of Π to G' is the transfer from G of an L-packet in the image of φ_H, we call Π *endoscopic*. If Π is not endoscopic then $d(\tau_f) = 3$ for all τ_f. If Π is endoscopic, then $d(\tau_f)$ is equal to 1 or 2. It may vary within Π_f and there is a simple locally-defined formula for $d(\tau_f)$ [**BRo**, **R**$_2$].

We now extend Π' to an L-packet Π on G'' with algebraic central character. The center of $G''(\mathbf{A}_F)$ is isomorphic to $\mathbf{A}_E^*$ and $G''(\mathbf{A}_F) = \mathbf{A}_E^* G'(\mathbf{A}_F)$. We obtain an extension Π by extending the central character of Π' to an algebraic Hecke character of $\mathbf{A}_E^*$. By a main result from [**M**], there is a compatible system $\rho(\tau_f) = \{\rho_\lambda\}$ of λ-adic representations of $\mathrm{Gal}(\overline{F}/F)$ associated to τ_f of dimension $d(\tau_f)$. Let $\chi(\tau_f)$ be the central character of τ_f. If $d(\tau_f) = 2$, then $\rho_\lambda \otimes \chi(\tau_f)^{-1}$ is unramified at almost all places v, including all finite primes v of relative degree 1 over F such that π'_v is unramified. For such v, we have

$$L_v(s, \rho_\lambda \otimes \chi(\tau_f)^{-1}) = L_v(s, \psi(\pi'))$$

(cf. [**BRo**, Theorem 1.9.1]). Reduction 1.2 follows, provided that we use the following lemma.

LEMMA. *There exists a choice of cuspidal representation π' of $U(\mathbf{A}_F)$ as in* §1.2 *such that $d(\tau_f) = 2$ for some $\tau_f \in \Pi_f$.*

Since $\psi(\pi')$ is cuspidal, the formula for $d(\tau_f)$ is given in terms of a function $\varepsilon \colon \Pi_f \to \{\pm 1\}$ which is defined locally: $\varepsilon(\tau_f) = \Pi \varepsilon(\tau_v)$ (product over the finite places) [**R**$_2$]. More precisely, there is a sign $\varepsilon(\pi''_\infty) = \pm 1$ depending only on π'_∞ such that $d(\tau_f) = 1$ if $\varepsilon(\tau_f) = \varepsilon(\pi''_\infty)$ and $d(\tau_f) = 2$ if $\varepsilon(\tau_f) \neq \varepsilon(\pi''_\infty)$. If Π_f contains more than one element, then ε maps onto $\{\pm 1\}$ and the Lemma is obvious. But if Π_f consists of a single representation τ_f, then $\varepsilon(\tau_f) = 1$ and we must show that there exists a choice of π' such that $\varepsilon(\pi''_\infty) = -1$. We can replace π' by a twist $\pi' \otimes \psi$ where ψ is a character of $U(1,1)$. Identify ψ with a character of $U(1)$ and suppose that $\psi_\infty(e^{i\theta}) = e^{in\theta}$. By [$\mathbf{R}_1$], §13.3, if $|n|$ is sufficiently large, $\varepsilon(\pi''_\infty) = -1$.

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