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TOEPLITZ C^*-ALGEBRAS OVER PSEUDOCONVEX REINHARDT DOMAINS

NORBERTO SALINAS, ALBERT SHEU AND HARALD UPMEIER

Multivariable Toeplitz operators, acting on Hardy or Bergman spaces over domains in $\mathbf{C}^n$, occur in connection with elliptic boundary value problems [**1**], weighted shift operators [**6**] and problems in function theory of several complex variables [**2**]. If the underlying domain is strictly pseudoconvex [**4**], of finite type [**1, 11**] or symmetric [**13**], the associated Toeplitz operators (with continuous symbol) are essentially commutative or at least generate a solvable C^*-algebra of finite length. In particular, the Toeplitz C^*-algebra is of type I.

In this note we describe the Toeplitz C^*-algebra of *pseudoconvex Reinhardt domains* Ω, using a finite composition series which is geometrically characterized by "boundary foliations" associated with the complex geometry of Ω. Whenever these foliations are of "irrational type," we obtain Toeplitz C^*-algebras which are not of type I (this can happen for domains with smooth boundary). We also announce an index theory for these non-type I Toeplitz C^*-algebras and give some applications to the theory of proper holomorphic mappings. For concreteness, we explain here the case $n = 2$.

Let Ω be a bounded pseudoconvex complete Reinhardt domain (in $\mathbf{C}^2$), with closure $\overline{\Omega}$. By [**8**], these domains are the natural domains of convergence of power series and are characterized by the condition that $(u, v) \in \Omega$ whenever $|u| \leq |z|$, $|v| \leq |w|$ for some $(z, w) \in \Omega$ or $|u| = |z_1|^\lambda |w_1|^{1-\lambda}$, $|v| = |z_2|^\lambda |w_2|^{1-\lambda}$ for some $(z_1, w_1) \in \Omega$, $(z_2, w_2) \in \Omega$ and $0 < \lambda < 1$. We may assume that Ω is *normalized*, i.e., Ω is contained in the bidisk $\mathbf{D}^2$ and contains the coordinate axes $V := \{(z, w) \in \mathbf{D}^2 \colon zw = 0\}$. Then the "logarithmic domain" $C := \{(x, y) \in \mathbf{R}^2 \colon (e^x, e^y) \in \Omega\}$ is an unbounded convex open set contained in the third quadrant and ∂C is a concave curve. Let $\overline{C}$ denote the closure of C in $\mathbf{R}^2$ and let $\partial^j(C)$ be the union of all j-dimensional *faces* of $\overline{C}$ (e.g., $\partial^2(C) = \overline{C}$ and $\partial^0(C)$ consists of all extreme points).

Given a face F of $\overline{C}$, denote by L_F the linear subspace of the same dimension parallel to F. For any point $t = (\xi, \eta)$ in the 2-torus $\mathbf{T}^2$, consider the leaf $t_F := \{(\xi e^{2\pi i x}, \eta e^{2\pi i y}) \colon (x, y) \in L_F\}$ generated by F through t. This gives a foliation $\mathscr{F}_F$ of $\mathbf{T}^2$, with corresponding foliation C^*-algebra (cf. [**5**]) denoted by $C^*(\mathscr{F}_F)$. For $F = \overline{C}$, $\mathscr{F}_F$ has just one leaf ($\mathbf{T}^2$ itself) and $C^*(\mathscr{F}_F)$ is *-isomorphic to the ideal $\mathscr{K}$ of compact operators. For

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$F = P$, an extreme point in $\partial C, \mathscr{F}_F$ is the trivial foliation where every point of $\mathbf{T}^2$ is a leaf, and $C^*(\mathscr{F}_F) \cong \mathscr{C}(\mathbf{T}^2)$. Here $\mathscr{C}(X)$ is the C^*-algebra of all continuous functions on a compact space X. If F is one-dimensional, $\mathscr{F}_F$ is the foliation of the Kronecker flow determined by the slope of F.

Let $H^2(\Omega)$ be the Bergman space of all (Lebesgue) square integrable holomorphic functions on Ω. Let $P\colon L^2(\Omega) \to H^2(\Omega)$ be the (orthogonal) Bergman projection. Then, for every $\varphi \in \mathscr{C}(\overline{\Omega})$, the bounded operator T_φ on $H^2(\Omega)$ defined by

$$T_\varphi(f) := P(\varphi f), \quad f \in H^2(\Omega)$$

is called the *Toeplitz operator* with symbol φ. The C^*-algebra generated by all these operators is denoted by $\mathscr{T}(\Omega)$.

THEOREM 1. *Let Ω be a (normalized) pseudoconvex complete Reinhardt domain in $\mathbf{C}^2$. Then the Toeplitz C^*-algebra $\mathscr{T}(\Omega)$ has a composition series $\mathscr{K} \subset \mathscr{I} \subset \mathscr{T}(\Omega)$, where $\mathscr{I}$ is the commutator ideal,*

$$\mathscr{T}(\Omega)/\mathscr{I} \cong \mathscr{C}(\partial^0(\Omega))$$

and

$$\mathscr{I}/\mathscr{K} \cong \sum_F^{\oplus} C^*(\mathscr{F}_F) \qquad (C^* \text{ -algebraic sum}).$$

Here $\partial^0(\Omega)$ is the closure (in $\mathbf{C}^2$) of the set $\{(\xi e^x, \eta e^y)\colon (\xi, \eta) \in \mathbf{T}^2, (x, y) \in \partial^0(C)\}$ and F runs over all 1-dimensional faces of $\overline{C}$.

If we let $\mathscr{I}_0 = 0$, $\mathscr{I}_1 = \mathscr{K}$, $\mathscr{I}_2 = \mathscr{I}$ and $\mathscr{I}_3 = \mathscr{T}(\Omega)$, we can uniformly state the conclusion of Theorem 1 as

$$\mathscr{I}_{j+1}/\mathscr{I}_j \cong \int_F^{\oplus} C^*(\mathscr{F}_F) \qquad (C^*\text{-direct integral}),$$

where F runs over all $(2-j)$-dimensional faces of $\overline{C}$, $0 \le j \le 2$.

COROLLARY 2. *$\mathscr{T}(\Omega)$ is of type* I *if and only if the slope of every 1-dimensional face in $\partial^1(C)$ is rational. Further, $\mathscr{T}(\Omega)$ is essentially abelian, i.e., $\mathscr{I} = \mathscr{K}$, if and only if there is no 1-dimensional face in ∂C, i.e., $\partial^1(C) = \varnothing$.*

The above results are proved in detail in **[12]**. The following purely geometrical result is a direct consequence of Corollary 2 and **[11**, Corollary 3.2].

COROLLARY 3. *Let Ω and Ω' be two normalized pseudoconvex complete Reinhardt domains. Let C and C' be the corresponding logarithmic domains, and assume there is a proper holomorphic mapping $\varphi\colon \Omega \to \Omega'$. If ∂C contains no 1-dimensional faces with irrational slope, then the same property holds for $\partial C'$. Further, if ∂C contains no 1-dimensional faces, then the same is true for $\partial C'$.*

Now we describe the index phenomenon in the presence of irrational slopes. We do this in the simplest nontrivial case, i.e., when Ω is the logarithmic convex hull of the union of two polydisks of multiradii $(\varepsilon, 1)$

and $(1,\delta)$, $\varepsilon < 1$, $\delta < 1$ (cf. [6]). Then the boundary of C consists of the line segment F joining $(\log\varepsilon, 0)$ and $(0, \log\delta)$ together with the negative part of both axes between $(-\infty, 0)$ and $(\log\varepsilon, 0)$ and between $(0, -\infty)$ and $(0, \log\delta)$. Assume that the corresponding slope $\beta = -\log\delta/\log\varepsilon$ is irrational. Then, as a consequence of Theorem 1, we have

$$\mathcal{I}/\mathcal{K} \cong [\mathcal{C}(\mathbf{T}) \otimes \mathcal{K}] \oplus [\mathcal{C}(\mathbf{T}) \otimes \mathcal{K}] \oplus C^*(\mathcal{F}_F)$$

and

$$\mathcal{T}(\Omega)/\mathcal{I} \cong \mathcal{C}(\mathbf{T}^2) \oplus \mathcal{C}(\mathbf{T}^2).$$

Let $\mathbf{Z}^2$ act on $\mathbf{R}$ by $\alpha(m, n; x) = x - n - m\beta^{-1}$, for $x \in \mathbf{R}$ and $(m, n) \in \mathbf{Z}^2$. The associated (strongly continuous) action of $\mathbf{Z}^2$ on $\mathcal{C}_0(\mathbf{R})$, again denoted by α, induces a *crossed product* C^*-algebra $\mathcal{C}_0(\mathbf{R}) \rtimes_\alpha \mathbf{Z}^2$ (defined as the C^*-completion of the convolution algebra of $\mathcal{C}_0(\mathbf{R})$-valued L^1-functions on $\mathbf{Z}^2$, cf. [9]), which is isomorphic to $C^*(\mathcal{F}_F)$ (not just stably isomorphic, cf. [10]). Further, we have $C^*(\mathcal{F}_F) \cong A_\beta \otimes \mathcal{K}$, where $A_\beta := \mathcal{C}(\mathbf{T}) \rtimes_\beta \mathbf{Z}$ is the *irrational rotation C^*-algebra* induced by the action of $\mathbf{Z}$ on $\mathbf{T}$ generated by the rotation with angle β. By Theorem 1, there is an ideal $\mathcal{I}_{\text{sing}} \subset \mathcal{I}$ containing $\mathcal{K}$ such that $\mathcal{I}_{\text{sing}}/\mathcal{K} \cong [\mathcal{C}(\mathbf{T}) \oplus \mathcal{C}(\mathbf{T})] \otimes \mathcal{K}$ and $\mathcal{I}/\mathcal{I}_{\text{sing}} \cong \mathcal{C}_0(\mathbf{R}) \rtimes_\alpha \mathbf{Z}^2$. The ideal $\mathcal{I}_{\text{sing}}$ induces an exact sequence

$$O \to \mathcal{I}/\mathcal{I}_{\text{sing}} \to \mathcal{T}(\Omega)/\mathcal{I}_{\text{sing}} \to \mathcal{T}(\Omega)/\mathcal{I} \to 0,$$

where $\mathcal{T}(\Omega)/\mathcal{I}_{\text{sing}} \cong \mathcal{C}(\mathbf{R} \cup \{\pm\infty\}) \rtimes_\alpha \mathbf{Z}^2$. Any short exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ of C^*-algebras has a topological invariant called the *index mapping* $\mathrm{Ind}\colon K_1(\mathcal{C}) \to K_0(\mathcal{A})$ on the level of K-theory (cf. [3]), which reduces to the ordinary (family) Fredholm index in case $\mathcal{A} = \mathcal{K}$ and $\mathcal{C}$ is commutative.

THEOREM 4. *The analytical index map*

$$\mathrm{Ind}\colon K^1(\mathbf{T}^2) \oplus K^1(\mathbf{T}^2) \to K_0(C^*(\mathcal{F}_F)),$$

associated with the above exact sequence (cf. [7]) *has the topological expression*

$$\mathrm{tr}(\mathrm{Ind}(\varphi \oplus \psi)) = \alpha(\mathrm{ch}(\varphi\psi^{-1}); 0) \quad \textit{for} \ \ \varphi, \psi \in K^1(\mathbf{T}^2),$$

where $\mathrm{tr}\colon K_0(C^*(\mathcal{F}_F)) \to \mathbf{R}$ *is the natural trace and*

$$\mathrm{ch}\colon K^1(\mathbf{T}^2) \to H^1(\mathbf{T}^2, \mathbf{Z}) \cong \mathbf{Z}^2$$

is the classical Chern character.

For the proof of the above theorem, see [12].

REMARK 5. We can easily construct a continuous function θ on $\overline{\Omega}$ such that the above index applied to the image of T_θ in $\mathcal{T}(\Omega)/\mathcal{I}$ yields a nonzero irrational number. For instance, let θ be any continuous function on $\overline{\Omega}$ such that $\theta(z, w) = w$ for $|z| = \varepsilon$, $|w| = 1$, and such that $\theta(z, w) = z$, for $|z| = 1$, $|w| = \delta$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66045