The infinite dimenisonal case is treated in Chapter 3. The author describes it as the "core of the book." The general saddle functional is introduced together with the necessary background in applied functional analysis. The theory is illuminated by model problems and examples.

Chapter 4 describes some extensions of the basic theory, and Chapter 5 is a wide-ranging account which it illustrates in the previous chapters in selected topics in the mechanics of fluids and elastic and plastic solids.

According to the author "the treatment is designed to be accessible in the first three chapters to final year undergraduates in mathematics and science" (in British Universities) which is just attainable with a careful selection of material. His second set of intended readers are "postgraduates and research workers in those subjects." The author has succeeded in the difficult task of achieving both these aims; he has made the subject very accessible and the book is a pleasure to read. It should interest researchers in a wide variety of applications of mathematics and particularly those who are looking for new approaches to seemingly intractable problems. It contains a wealth of representative examples and applications which the reader can dip into, and study and develop for his or her own particular interests.

References

1. B. Noble and M. J. Sewell, *On dual extremum principles in applied mathematics*, J. Inst. Math. Appl. **9** (1972), 123–193.

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Torsion theories, by Jonathan S. Golan. Longman Scientific and Technical, Essex, and John Wiley and Sons, New York, 1986, xviii + 651 pp., $175.00. ISBN 0-582-99808-5

Torsion Theory traces its origins back to two independent developments in the 1950s. On the one hand was the generalized theory of localization of noncommutative rings being worked out by Johnson, Utumi, Lambek and others. On the other hand was the theory of localization of categories originated by Serre and first formalized by Grothendieck in [**2**]. It was not long before the connections between these two ideas were noticed and the theory synthesized, the most notable contribution here being that of Gabriel [**3**]. By the early 1970s this generalized theory of localization of rings and categories had reached a fairly mature stage and a number of good accounts of the subject appeared (for instance, [**6**, **7**]).

For those unfamiliar with the subject, the idea is the following. A torsion class is a full subcategory of the category R-Mod of all modules over the ring R, which is closed under submodules, homomorphic images, extensions and direct sums. Given a torsion class $\mathscr{T}$, we may then form a quotient category R-Mod $/\mathscr{T}$ and a localization functor from R-Mod to R-Mod $/\mathscr{T}$. When $\mathscr{T}$ is suitably nice (for instance in the case of classical Ore localizations), the category R-Mod $/\mathscr{T}$ is precisely the category of modules over the localized ring $R_{\mathscr{T}}$. Such localizations are called perfect localizations. In general, however, this will not be the case. Nonclassical torsion theoretic localization occurs in a number of interesting 'natural' situations. The canonical example is the following. Let V be an affine elliptic curve over $\mathbf{C}$ and let U be the complement in V of a point with infinite order (in the divisor class group). Then the coordinate ring $\mathbf{C}[U]$ is a nonclassical perfect localization of $\mathbf{C}[V]$. For a noncommutative example, let D (resp. A) be the ring of global differential operators on complex projective (resp. affine) n-space. Then A is just the nth Weyl algebra and D is a primitive factor of the enveloping algebra of the Lie algebra $\mathfrak{sl}\,(n+1, \mathbf{C})$. It is an almost immediate consequence of the famous Bernstein-Beilinson Theorem **[1]** that the restriction map of D into A is a perfect localization. Since the units of both rings are just the nonzero scalars, this localization is certainly nonclassical. As a final example, let R be an hereditary order contained in the maximal order A. Then A is again a perfect nonclassical localization of R. For a neat application of this viewpoint, the reader is referred to Keating's calculation of the K-theory of tiled orders **[5]**.

One does not have to look much further for an example of a nonperfect localization. Let $R = \mathbf{C}[x, y]$ and let $\mathscr{T}$ be the class of R-modules supported at the origin. Then $\mathscr{T}$ is a torsion class and the corresponding localization $R_{\mathscr{T}}$ is isomorphic to R. Thus the quotient category R-Mod $/\mathscr{T}$ is certainly not naturally isomorphic to $R_{\mathscr{T}}$-Mod. However the quotient category is equivalent to the category of coherent modules over the quasi-affine variety $A^2(C)\backslash(0, 0)$. Thus the theory of schemes is of great help in understanding such localization in the commutative situation. In the noncommutative case, there is no analogous theory, and the localized category is much harder to work with. This is perhaps the most serious weakness of the general theory and accounts to some extent for the direction the subject is currently headed in.

In recent years researchers in torsion theory have moved away from localization towards a study of module categories in terms of torsion theories. This work has a number of different foci. Probably the largest area in terms of quantity of research is the subject which can perhaps fairly be called relative module theory. Let τ be a torsion theory defined by a torsion class $\mathscr{T}$. (The distinction between a torsion theory and a torsion class is not important here; suffice it to say that localization with respect to τ is the same as the localization described earlier with respect to the class $\mathscr{T}$ of τ-torsion modules.) Almost any of the standard definitions of module theory may be generalized by weakening the condition that a module be zero (or in some cases strengthening the condition that a module be arbitrary) to the condition that the module be τ-torsion. (Of course, this is not done arbitrarily.) For example, a module M is defined to be τ-artinian if for all descending chains of submodules, almost all the factors are τ-torsion.

On the other hand a τ-projective module is a module which is projective with respect to all epimorphisms whose kernel is τ-torsion. Given these new definitions, one may ask whether there are relative versions of the standard results of module theory—the possibilities are clearly enormous. Some of the proofs of these results are fairly straightforward analogues of the original result while others involve considerable technical difficulties. As the first 200 pp. of Golan's book chronicle, much progress has been made in this direction. Thus, for instance, there are relative versions of the Hopkins-Levitski theorem, the Jordan-Holder theorem, the characterization of Noetherian rings in terms of injective modules and many other familiar results. Given the nature of this subject, it is natural to ask how much further this program can or should be taken. I was disappointed that no perspective on this question was offered by the author.

A second significant area is the study of the torsion theories themselves from both a global and a local point of view. Meet and join of torsion theories may be defined in such a way that the set of all torsion theories forms a (complete Brouwerian) lattice. The concept of prime torsion theory can be defined and attempts have been made to introduce a noncommutative algebraic geometry using a notion of torsion-theoretic spectrum. Due to the technical complications involved in the noncommutative situation, this approach has met with only limited success. Special types of torsion theories are studied in great detail, for instance the stable and Jansian torsion theories (for which the class of torsion modules is closed under essential extensions and direct products, respectively). Much more detailed structural information may be obtained for these special torsion theories and certain types of rings (such as semiartinian rings) may be characterized by properties of the torsion theories over the ring. In order to understand the complications and difficulties arising in the study of torsion theories, it is enough to look at some well-known classes of rings, say Weyl algebras or enveloping algebras of Lie algebras, and ask (a) what can be said about the torsion theories over these rings and (b) how torsion theory can help us understand these rings. The answer to both questions is still, unfortunately, 'very little'. This situation accounts, at least in part, for the deepening gulf between torsion-theorists and other algebraists.

Professor Golan's book is an updated version of his earlier book *Localization of Noetherian rings* [**4**]. The material covered is similar but the basic approach has been changed to reflect the current viewpoint on the subject. In particular, localization is not introduced until p. 250 and quotient categories are never discussed. Although essentially self-contained modulo a basic knowledge of rings and modules, the book is structurally more of an encyclopedia of torsion theory than an introductory text. It is well organized and clearly written and should serve as an indispensable reference for researchers in the field. A typical chapter begins with a definition of, for instance, a relatively injective module or a symmetric torsion theory and then goes on to develop the principal results pertaining to this definition. However, this book would not serve well as an introductory text for the uninitiated student. The dry style of definition, lemma, proposition gives the student almost no explanation or motivation. For example, a short discussion of why the definition of τ-projective should be as it is, would be of far greater use to the reader than the proof given of the Relativized Schanuel's Lemma which is more or less a word-for-word repetition of the standard proof. There are other stylistic problems with the book. An unusually large number of results in this book are of the form: "Let τ be a ____ torsion theory on R-mod. Then the following conditions are equivalent: ... ". It is not uncommon for 7 or more conditions to follow. For instance 9 equivalent conditions are given for a Jansian torsion theory to be centrally splitting while a massive 18 equivalent conditions are given for a Jansian torsion theory to be stable. Such emphasis on this kind of result makes the material extremely difficult to absorb and more significantly gives an impression of superficiality to the whole subject. A reasonable number of relevant examples and counter-examples are sprinkled through the text. However they often appeared to be constructed specifically to illustrate the theory. Very few of the examples presented convinced me that the machinery being built up was going to prove useful in the study of 'naturally occurring' rings or to be of interest to other ring theorists.

Given the price tag of $175 and the limitations described above I would not recommend this book to any but experts in the field. But then this is, I believe, the audience for whom the book was intended.

REFERENCES

1. J. Bernstein and A. Beilinson, *Localisation de $\mathfrak{g}$-modules*, C. R. Acad. Sci. Paris **292** (1981), 15–18.

2. A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) **9** (1957), 119–221.

3. P. Gabriel, *Des catégories abeliennes*, Bull. Soc. Math. France **90** (1962), 323–448.

4. J. Golan, *Localization of noncommutative rings*, Marcel Dekker, New York, 1975.

5. M. E. Keating, *On the K-theory of tiled orders*, J. Algebra **43** (1976) 193–197.

6. N. Popescu, *Abelian categories with applications to rings and modules*, Academic Press, London, 1973.

7. B. Stenstrom, *Rings of quotients*, Springer-Verlag, Berlin and New York, 1975.

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