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Maximum and minimum principles: A unified approach, with applications, by M. J. Sewell. Cambridge University Press, Cambridge, New York, Melbourne, 1987, xvi + 468 pp., $79.50 (cloth), $34.50 (paper). ISBN 0-521-33244-3

It seems likely that every system of equations can be said to give the location of the stationary value of some function: all we need to justify this conjecture is a suitable function space into which the system can be embedded, an appropriate gradient operator and the function! On the one hand this could be a trivial exercise of no great significance, whilst, alternatively, it might involve an elaborate construction in functional analysis. Self-evidently there will be no unique trio of space, gradient and function which will achieve this stationary principle. For many applications there are 'natural' variational formulations based on energy considerations which lead to stationarity principles. However, there are many differential equations which model nonlinear, dissipative systems which have no obvious variational context.

The construction of stationary principles can often be achieved without too much technical difficulty, but it is generally not enough. From the standpoint of applications in Physics and Engineering, further demands are usually required of the construction. In many problems the generating functional is an integral and the application will only become practically interesting if the so-called stationary value is of some physical significance. Further, there may be side constraints to be satisfied; for example, these could be initial or boundary conditions, integral or differential constraints, or inequalities. Even with a complete system, a variational principle which yields only stationarity may be of limited practical value although the principle itself may indicate the fundamental processes at work in the particular application. To achieve a minimum principle, and, consequently, an upper bound for the stationary value we require further structure in the generating functional. Local convexity of the functional in the neighbourhood of its stationary value provides a sufficient condition for a minimum principle; it is a bonus if we can design functionals which have global convexity over a convenient linear space.

Obviously from an estimating point of view, a minimum principle is a one-sided bound with limited practical use. Ideally, both a maximum and a minimum principle are required for numerical estimation purposes, and this is where the notion of the saddle functional plays an important role. Usually the functional is defined on a product of two linear spaces in such a manner that it is convex in one space, concave in the other and stationary where the dual gradients vanish. The advantage of the convex/concave saddle is that certain partial gradient solutions provide upper bounds and others supply lower bounds to the stationary value.

The construction of bounds is not the only purpose of this approach to variational principles. The origins of this subject lie in duality and the

Legendre transformation which has close links with physical theories in compressible fluid flow, plasticity and thermodynamics. This idea stems from the notion that any functional of two variables can be used to generate three other Legendre dual functionals by holding either variable fixed or neither one fixed. This theory has also had more recent application in the classification of singularities and bifurcations.

There are many techniques which have been developed from the basic saddle problem including asymptotic, comparison and iterative methods and pointwise estimates. And yet the subject never quite seems to live up to its promise. Variational calculus is very "open" in the sense that almost any problem can be cast as a stationary variational problem, and, with a little ingenuity, also as a convex or saddle extremum. But does this help us to actually *solve* differential and integral equations, for example? The development of general iterative, convergent sequences of upper and lower bounding principles seems some way off at present. On the other hand, from a strictly utilitarian point of view, some sort of application using, for example, the Rayleigh-Ritz procedure can usually be put together when all else fails. Perhaps the general accessibility of convex variational calculus is its attractive feature.

Sewell's book on *Maximum and minimum principles* reflects the author's interest in the subject spanning some 25 years. Whilst the building blocks of the subject, such as the calculus of variations, convexity and functional differentiation, belong to earlier eras, much of this modern treatment of the subject was developed initially and independently by Ben Noble and the author of this book, and later in collaboration. A survey of their early work can be found in [**1**]. The saddle quantity S is defined by [see p. 216 of the book]

$$S = L_+ - L_- - \left(x_+ - x_-, \left.\frac{\partial L}{\partial x}\right|_+\right) - \left\langle u_+ - u_-, \left.\frac{\partial L}{\partial u}\right|_-\right\rangle$$

where $L[x, u]: X \times U \to \mathbf{R}$, X and U are inner product spaces with inner products $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ and x_+, u_+ and x_-, u_- are pairs of points in the infinite dimensional domain of the functional $L[x, u]$. In the definition of S, $\partial L/\partial x$ and $\partial L/\partial u$ are partial gradients of L. The functional $L[x, u]$ is then said to be a strict or weak saddle functional over some subdomain if $S > 0$ or $S \geq 0$ respectively. In their simplest form, dual extremum principles follow by choosing a plus pair to satisfy $\partial L/\partial x = 0$ and a minus pair to satisfy $\partial L/\partial u = 0$. These partial solutions then imply that $L_+ > L_-$ and that the stationary value is trapped between L_+ and L_-. There are many variants of these principles described in the book but this definition contains the essential idea.

The book is divided into five long chapters. Chapter 1 looks at the finite dimensional case by way of an introduction to the subject starting with the geometry of the saddle surface. The main idea of the upper and lower bounding principles are developed; many detailed examples are included.

Chapter 2 gives a complete account of the Legendre transformation. This book is probably the first text to integrate singularity theory and bifurcation, or as it is more generally called catastrophe theory, into duality.

The infinite dimenisonal case is treated in Chapter 3. The author describes it as the "core of the book." The general saddle functional is introduced together with the necessary background in applied functional analysis. The theory is illuminated by model problems and examples.

Chapter 4 describes some extensions of the basic theory, and Chapter 5 is a wide-ranging account which it illustrates in the previous chapters in selected topics in the mechanics of fluids and elastic and plastic solids.

According to the author "the treatment is designed to be accessible in the first three chapters to final year undergraduates in mathematics and science" (in British Universities) which is just attainable with a careful selection of material. His second set of intended readers are "postgraduates and research workers in those subjects." The author has succeeded in the difficult task of achieving both these aims; he has made the subject very accessible and the book is a pleasure to read. It should interest researchers in a wide variety of applications of mathematics and particularly those who are looking for new approaches to seemingly intractable problems. It contains a wealth of representative examples and applications which the reader can dip into, and study and develop for his or her own particular interests.

REFERENCES

1. B. Noble and M. J. Sewell, *On dual extremum principles in applied mathematics*, J. Inst. Math. Appl. **9** (1972), 123–193.

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Torsion theories, by Jonathan S. Golan. Longman Scientific and Technical, Essex, and John Wiley and Sons, New York, 1986, xviii + 651 pp., $175.00. ISBN 0-582-99808-5

Torsion Theory traces its origins back to two independent developments in the 1950s. On the one hand was the generalized theory of localization of noncommutative rings being worked out by Johnson, Utumi, Lambek and others. On the other hand was the theory of localization of categories originated by Serre and first formalized by Grothendieck in **[2]**. It was not long before the connections between these two ideas were noticed and the theory synthesized, the most notable contribution here being that of Gabriel **[3]**. By the early 1970s this generalized theory of localization of rings and categories had reached a fairly mature stage and a number of good accounts of the subject appeared (for instance, **[6, 7]**).

For those unfamiliar with the subject, the idea is the following. A torsion class is a full subcategory of the category R-Mod of all modules over