But these are minor matters. This monograph is a welcome addition to the list of books to which one can send people who want to learn about modern real analysis.

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Einstein manifolds, by Arthur L. Besse. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 10, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1987, xii + 510 pp., $89.00. ISBN 3-540-15279-2

Even when speaking to a group of differential geometers one cannot safely assume that everyone knows what an Einstein metric is. Why then would A. L. Besse write a 500 page book on the subject? With characteristic frankness he addresses that question in §B of the excellent nineteen page introduction (an impressive mathematical essay in its own right). This book is his response to the importunities received over his career to write a treatise on Riemannian geometry. This is the treatise, but with a focus and sense of purpose that make it far more exciting and fun than most weighty treatises ever are. Using Einstein spaces as his reference point, he can easily and naturally enter into some of the most exciting research activity of today: topology (the Poincaré conjecture), partial differential equations (Aubin-Yau solution of the Calabi problem, Yang-Mills theory), and the wonderful world of four dimensional geometry (self-duality and the Penrose construction), to name just three. In the final analysis, however, the author has a passion for Einstein metrics in their own right and his intention is to teach the reader what he knows about them, what great number of unanswered questions can be naturally asked about them, and why they merit enthusiastic study.

What then is an Einstein metric? A Riemannian metric g on an n-dimensional manifold M is a collection of positive definite inner products on the tangent spaces of M, one at each point, which vary smoothly in the sense that the inner product of any pair of C^∞ vector fields on M is a C^∞ function on M. With respect to local coordinates $x^1, \dots, x^n$ in M, $g = \sum g_{ij}\, dx^i\, dx^j$, where $g_{ij} = g_{ji}$ are C^∞ functions. For example, in Euclidean space $\mathbf{R}^n$ where every tangent space is identified with $\mathbf{R}^n$ itself, the canonical Riemannian metric is $\sum (dx^i)^2$, the standard inner product on $\mathbf{R}^n$. In his inaugural address Riemann showed that his curvature tensor R (n^4 local functions R_{ijkl} with respect to local coordinates) provides the local criteria as to when there exists a local coordinate system with respect to which $g = \sum(dx^i)^2$. Namely, this is possible if and only if $R = 0$, in which case we say that g is flat. Being a tensor, $R_{ijkl} = 0$ with respect to some coordinate system if and only if they are zero with respect to any local coordinate system.

The big problem with Riemann's curvature tensor is its enormous number of components R_{ijkl}. Some relief is provided by the familiar symmetries $R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}$ (which mean that at a point p in M, R is a symmetric linear transformation on the space of skew-symmetric 2-tensors at p) together with the Bianchi identities $R_{ijkl} + R_{iljk} + R_{iklj} = 0$. This still leaves $n^2(n-1)(n+1)/12$ components (see §1.108). When n is 2 the one component is R_{1212}, which is related to the Gaussian curvature K by $K = R_{1212}/\det g$, where $\det g = g_{11}g_{22} - g_{12}^2$. That's encouraging, but when $n = 3$ there are already six components, and when $n = 4$ there are 20.

Contraction of a tensor (taking its trace) produces a tensor with two fewer indices. The only nontrivial way (up to sign) to contract R is by its first and third indices, which produces the Ricci tensor r, in local coordinates $r_{ij} = \sum g^{kl}R_{kilj} = r_{ji}$, a symmetric 2-tensor, thus of the same type as the metric g itself (here g^{ij} is the inverse matrix of g_{ij}). G. C. Ricci introduced his tensor as a possible abstract substitute for the second fundamental form of a hypersurface, an apparently fruitless aspiration (see Introduction §H), but nevertheless it has turned out to be a tensor with far-reaching applications in geometry and relativity. Repeating the contraction process one obtains the scalar curvature $s = \sum g^{ij}r_{ij}$, a function on M, independent of choice of local coordinate system.

Coming finally to the point, g is an Einstein metric if $r = \lambda g$ for some constant λ, called the Einstein constant. When $n = 2$ it is always the case that $r = Kg$, so the Einstein condition here means the Gaussian curvature K is constant. If $n > 2$, then the condition $r = \lambda g$ for λ any function on M, always implies that λ is constant. Over the past century the most frequently used expression of R has been through its sectional curvatures (because they appear in the second variation of length-minimizing curves) which are defined as follows. Let a 2-dimensional subspace Π of the tangent space T_pM have an orthonormal basis v, w. Then the sectional curvature of the plane (p, Π) is $R(v, w, v, w)$. By an elementary calculation, if $n = 3$, then g is Einstein if and only if g has constant sectional curvature, i.e., the same sectional curvature on all tangent 2-planes (p, Π).

The famous, still unsolved, Poincaré conjucture is the name given to the question posed by Poincaré: Is the 3-sphere the only compact simply connected 3-manifold? That is certainly true in 2 dimensions. In any introductory course on Riemannian geometry it is proved that if a compact simply connected n-manifold has a Riemannian metric of constant positive sectional curvature K, then it is the n-sphere S^n with its canonical metric inherited from $\mathbf{R}^{n+1}$ in which it sits as the sphere of radius $1/\sqrt{K}$. Thus if we can prove that any compact simply connected 3-manifold possesses an Einstein metric, then we have proved the Poincaré conjecture.

From the point of view of partial differential equations this idea is not so far fetched (see Chapter 5, §G). R. Hamilton used the Ricci tensor of a given metric g in order to deform it to an Einstein metric. Considering the heat type equation $\partial g/\partial t = 2\rho g/n - 2r_g$, where ρ is a constant equal to the average of the scalar curvature s of g, he proves that if g is a metric on the compact connected 3-manifold M such that r_g is positive definite at every point, then the solution g_t of the above equation goes to an Einstein metric as $t \to \infty$. About ten years earlier, T. Aubin had proved that if $r_g \geq 0$ at every point of M, and is positive definite at some point, then there exists a Riemannian metric on M with positive definite Ricci curvature everywhere. So the approach gets intriguingly close, but ... still no cigar.

Einstein Manifolds is not one of those expository surveys which tells you about a great variety of results, but sends you to the research literature for any details, proofs or real understanding. Topics are covered in detail with proofs and commentary, what might be called annotated proofs. Five pages of Chapter 5, §G, are devoted to Hamilton's proof.

Any compact surface admits a metric of constant Gaussian curvature. If a compact n-manifold admits a metric of constant sectional curvature, then its universal cover is diffeomorphic to $\mathbf{R}^n$ or the n-sphere. Thus $S^2 \times S^1$ does not admit a metric of constant sectional curvature. Do you know where to find a good brief account of compact 3-manifolds of constant negative curvature? Try Chapter 6, §C. One point made is that constant sectional curvature is not the correct generalization for higher dimensions of constant Gaussian curvature for 2-manifolds. It is too strong. Einstein metrics may be the correct generalization.

In dimension 4 we find Einstein metrics which do not have constant sectional curvature, for example the complex projective plane $\mathbf{CP}^2$ with its Fubini-Study metric (see §9.77). The sectional curvatures of this metric fill out the interval $[1,4]$ at every point. In addition, this manifold is complex and this metric is Kaehlerian. It is an example of the important class of Kaehler-Einstein spaces (see Chapter 2). The Ricci tensor of Kaehler spaces gives rise to a closed type (1,1) form, the Ricci form of course, which represents the first Chern class of the manifold.

This topological property of the Ricci tensor on a Kaehler manifold is very important. Let g and h be Kaehler metrics on the complex manifold M, and let ρ_g and ρ_h denote their Ricci forms. Then their volume forms are related by $\mu_g = f\mu_h$, where $f > 0$ everywhere, and the Ricci forms satisfy

$$\rho_g = \rho_h - \mathrm{i}d'\, d'' \log f. \tag{1}$$

The famous Calabi problem is, given g and $f > 0$, does there exists a Kaehler metric g on M whose Ricci form is given by (1)? The answer (due to S. T. Yau and T. Aubin) is yes if M is compact (see §2.101 and Chapter 11). Solution of this problem involved deep and significant research into PDE of Monge-Ampère type. Thus it is that the study of Einstein metrics leads into significant problems, this time in PDE.

The Einstein space approach to the Poincaré conjecture is based on the supposition that there are no topological restrictions imposed by the existence of an Einstein metric on a compact simply connected 3-manifold (except that the sign of the Einstein constant must be positive). In dimension 4 this is not the case, while in dimensions greater than 4 no such topological restrictions are known. The result for 4-manifolds, due to N. Hitchin and J. Thorpe (see §6.35), is that the Euler characteristic χ and signature τ of a compact oriented 4-dimensional Einstein manifold must satisfy the inequality $\chi \geq \frac{3}{2}|\tau|$. This result follows from the important role played by Einstein metrics in the wonderful world of 4-dimensional Riemannian geometry (see §6D), where the Hodge $*$ operator creates the beautiful properties of self-duality, and where we meet the marvelous Penrose twistor spaces (see Chapter 13).

This material, by the way, is given a masterly presentation in §§1G and 1H and in Chapter 13. The book was written to be read piecemeal. (Who really reads mathematics any other way?) The author does not hesitate to recapitulate on p. 370 what he wrote on p. 51 if he thinks it will help the reader who has just jumped into Chapter 13. Needless to say, this kind of style is essential for a really useful reference book.

Their name suggests that Einstein metrics have something to do with general relativity. Most of the relationship is analogy, but analogy is important and the author includes Chapter 3 on Relativity to explain it all. Other physics in which Einstein spaces play a more direct role includes Kaluza-Klein theory (see §9.63) and Yang-Mills theory (see §9.35). These arise in the midst of a general discourse on Riemannian submersions (Chapter 9) applied to special homogeneous Riemannian manifolds (Chapter 7). From this discourse come most of the known examples of Einstein spaces.

Einstein Manifolds represents a group effort of the sort which must be unique in the mathematical world. In an Acknowledgement, A. Besse lists and thanks his friends who helped write the book. It is a grand group effort created in conditions of anonymity which most mathematicians could not, or would not, ever be able to afford. The result is a high quality Springer-Verlag Ergebnisse series book, with a breadth of view beyond most individuals' capacity, well-organized and carefully planned down to the finest details of notation and paragraph numbering. There is a 21 pp. bibliography, a notation index and a subject index. There is an appendix on Sobolev spaces and elliptic operators, and lest the reader fear that material appears in a book only after it is mined dry, there is an eight page addendum with last minute results on Einstein metrics.

One of these results is a set of examples of compact manifolds M, e.g., $S^2 \times S^3$, which possess an infinite number of distinct Einstein metrics $(g_i)_1^\infty$, such that after the appropriate normalization, the Einstein constants λ_i (in the Einstein equations $r_i = \lambda_i g_i$) tend to zero as $i \to \infty$. Einstein metrics are such an appealing subject of study, not only because they involve themselves in so many important fields of mathematics, but in no small part because even though A. Besse and his friends have just published this excellent 500 page book about them, so little is actually known that examples remain the central object of research in the subject. Indeed, the author offers (see §0.23) dinner at a starred French restaurant to anyone who sends him an example not in his book of a Ricci flat compact manifold ($r = 0$). Give it a try, and then read his book.

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