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Lectures on Bochner-Riesz means, by Katherine Michelle Davis and Yang-Chun Chang. London Mathematical Society Lecture Notes, Vol. 114, Cambridge University Press, Cambridge, New York, Melbourne, 1987, ix + 150 pp., $22.95. ISBN 0-521-31277-9

"The theory of functions of several real variables": sounds old-fashioned, doesn't it? But look in your copy of Hewitt and Stromberg [2] or Royden [3]. You'll find plenty of analysis on the real line, and plenty of analysis on abstract topological spaces and measure spaces, and not much in between. This gap is partly a reflection of the temper of the times; but in the early 1960s when these books were written, beyond the level of calculus there really *wasn't* much in between, at least not much that was ready to be transplanted from research journals to books.

In the intervening quarter-century the situation has changed enormously, and analysis on $\mathbf{R}^n$ is now a thriving subject. Among the main lines of development are the following:

(1) Banach spaces of functions and generalized functions defined in terms of various growth or smoothness conditions: L^p spaces, Hardy spaces, Sobolev spaces and their relatives, BMO, and so forth. Closely intertwined with this are the theory of differentiability and the study of approximation of arbitrary functions by suitable types of smooth functions, such as trigonometric polynomials or harmonic functions.

(2) Singular integral operators, oscillatory integral operators, and operators defined by convolutions or Fourier multipliers, and their continuity properties with respect to the Banach spaces mentioned above. These classes of operators include pseudodifferential operators and their generalizations, as well as the Fourier transform.

(3) Nonlinear operators such as maximal functions, Littlewood-Paley g-functions, and Lusin area integrals that can be used to control the behavior of functions in various ways.

The motivation for all of this comes from several sources. Much of it is in response to questions that arise in partial differential equations, several complex variables, and other branches of analysis. Also, there is an obvious desire to extend the sort of detailed understanding that was achieved for functions of one variable in the first half of this century to higher dimensions. And of course, once a subject like this gets going, it generates its own ideas and techniques, which often turn out to have applications in unexpected places.

After these generalities, let us come down to earth by contemplating a very concrete, very basic problem: the L^p convergence of Fourier series. Let Q denote the unit cube $[0, 1]^n$ in $\mathbf{R}^n$. If f is a function on $\mathbf{R}^n$, periodic with period 1 in each variable and integrable on Q, we can form its Fourier coefficients

$$\hat{f}(k) = \int_Q f(x) e^{-2\pi i k\cdot x}\, dx, \qquad k = (k_1, \dots, k_n) \in \mathbf{Z}^n,$$

and thence its Fourier series

$$\sum_{k\in\mathbf{Z}^n} \hat{f}(k) e^{2\pi i k\cdot x}. \tag{1}$$

The series (1) always converges to f in some weak sense, and we can ask whether it converges to f in the L^p norm when $f \in L^p(Q)$. To make this precise, we interpret (1) as the limit of its spherical partial sums,

$$S_R f(x) = \sum_{|k|<R} \hat{f}(k) e^{2\pi i k\cdot x} \qquad \left(|k|^2 = \sum k_j^2\right),$$

(there are other interpretations of (1), leading to other results!), and we ask

Is it true that $\lim\limits_{R\to\infty} \|S_R f - f\|_p = 0$ whenever $f \in L^p(Q)$?

For $p = 2$ the answer is always *yes*, because the functions $e^{2\pi i k\cdot x}$ form an orthonormal basis for L^2. On the other hand, for $p = \infty$ (and hence, by a duality argument, for $p = 1$) the answer is always *no*: it has been known for over a century that the Fourier series of a continuous periodic function f need not converge uniformly, or even pointwise, to f. In the one-dimensional case, an old theorem of Marcel Riesz assures us that the answer is *yes* for $1 < p < \infty$. It therefore comes as an unpleasant surprise to discover, as Charles Fefferman did in 1971, that when $n > 1$, the answer is *no* for all $p \neq 2$.

What to do? Well, it was recognized long ago that one-dimensional Fourier series can be made more tractable by employing summability methods that cut off the Fourier coefficients in some smoother way than simply taking partial sums. For example, the Fourier series of a continuous f is always uniformly Cesàro or Abel summable to f. A convenient summability method in higher dimensions that is closely related to the spherical partial sums $S_R f$ is provided by the *Bochner-Riesz means* of order $\alpha > 0$, defined by

$$S_R^\alpha f(x) = \sum_{|k|<R} (1 - R^{-2}|k|^2)^\alpha \hat{f}(k) e^{2\pi i k\cdot x}.$$

(As $\alpha \to 0$ these become the partial sums of (1), and when $n = 1$ and $\alpha = \frac{1}{2}$, they are essentially equivalent to the classical Cesàro means.) We now ask

$$\text{Is it true that } \lim_{R\to\infty} \|S_R^\alpha f - f\|_p = 0 \quad \text{whenever } f \in L^p(Q)?$$

It is not hard to show that the answer is *yes* for all p, even $p = 1$ and (when f is continuous) $p = \infty$, provided that $\alpha > (n-1)/2$. For smaller values of α, the results are deeper and more recent. It is known that the answer is *yes* when

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1+2\alpha}{2n} \tag{2}$$

provided that either $n = 2$ or $\alpha > (n-1)/2(n+1)$, and *no* when (2) fails. The situation when α is near zero and $n > 2$ is still not completely understood.

The nominal aim of the book under review is to prove the theorems outlined above, both the negative result for S_R and the positive results for S_R^α. But in fact, the book does much more: it could well have been subtitled "A working introduction to modern Fourier analysis." The authors start more or less from scratch, i.e., from a background in analysis such as can be found in **[2]** or **[3]**, and develop the machinery they need as they go along. They take pains to explain the conceptual framework underlying the methods and results and to give the intuitive motivation behind technical arguments. As a result, one who reads this book from cover to cover will not only learn how to sum multiple Fourier series, but will become conversant with many of the most important current tools of the real analyst, such as Cotlar's lemma, maximal functions, "good λ" inequalities, weights of class A_p, the multiplier theorems of Mihlin-Hörmander and Marcinkiewicz, restriction theorems, and Meyer's lemma.

The authors employ a determinedly lively and informal style that will undoubtedly be appreciated by readers struggling to understand the technical material. Sometimes they overdo it (I think that calling the standard limiting process for defining the Fourier transform on L^2 "trickery and deceit" is an abuse of poetic license), but that is still preferable to an excessively dry presentation.

The book suffers a bit from occasional sloppiness in content and typography. A few examples: On p. 2 the authors describe the theorems about S_R^α which they want to prove by drawing two diagrams, but they forget to explain that the first and second diagrams pertain to the cases $n = 2$ and $n > 2$ respectively. The paper **[1]** of Carleson and Sjölin is referred to in the text, but it is omitted from the bibliography. And there is no excuse for letting a TEXnical typist get away with things like "$isign(\xi)$" when "$i\,\mathrm{sign}(\xi)$" is just as easy to produce.

But these are minor matters. This monograph is a welcome addition to the list of books to which one can send people who want to learn about modern real analysis.

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Einstein manifolds, by Arthur L. Besse. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 10, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1987, xii + 510 pp., $89.00. ISBN 3-540-15279-2

Even when speaking to a group of differential geometers one cannot safely assume that everyone knows what an Einstein metric is. Why then would A. L. Besse write a 500 page book on the subject? With characteristic frankness he addresses that question in §B of the excellent nineteen page introduction (an impressive mathematical essay in its own right). This book is his response to the importunities received over his career to write a treatise on Riemannian geometry. This is the treatise, but with a focus and sense of purpose that make it far more exciting and fun than most weighty treatises ever are. Using Einstein spaces as his reference point, he can easily and naturally enter into some of the most exciting research activity of today: topology (the Poincaré conjecture), partial differential equations (Aubin-Yau solution of the Calabi problem, Yang-Mills theory), and the wonderful world of four dimensional geometry (self-duality and the Penrose construction), to name just three. In the final analysis, however, the author has a passion for Einstein metrics in their own right and his intention is to teach the reader what he knows about them, what great number of unanswered questions can be naturally asked about them, and why they merit enthusiastic study.

What then is an Einstein metric? A Riemannian metric g on an n-dimensional manifold M is a collection of positive definite inner products on the tangent spaces of M, one at each point, which vary smoothly in the sense that the inner product of any pair of C^∞ vector fields on M is a C^∞ function on M. With respect to local coordinates $x^1, \dots, x^n$ in M, $g = \sum g_{ij}\, dx^i\, dx^j$, where $g_{ij} = g_{ji}$ are C^∞ functions. For example, in Euclidean space $\mathbf{R}^n$ where every tangent space is identified with $\mathbf{R}^n$ itself, the canonical Riemannian metric is $\sum (dx^i)^2$, the standard inner product on $\mathbf{R}^n$. In his inaugural address Riemann showed that his curvature tensor R (n^4 local functions R_{ijkl} with respect to local coordinates) provides the