but rather with novel applications of the tools of the Malliavin calculus. While the applications using the Malliavin derivative (already discussed) are not mentioned by Bell, he does nevertheless present diverse applications in Chapter 7, including such disparate subjects as filtering theory and infinite particle systems. Here he could be a bit more authoritative: For example, in the filtering theory section he should mention further work, at least at the bibliographic level (e.g., [**1**, **2** and **5**]).

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A course in constructive algebra, by Ray Mines, Fred Richman, and Wim Ruitenburg. Universitext, Springer-Verlag, New York, Berlin, Heidelberg, xi + 344 pp., $32.00. ISBN 0-387-96640-4

Is every ideal J in the ring $\mathbf{Z}$ of integers principal?—that is, given an ideal J of $\mathbf{Z}$, can we find an integer m—called a generator of J—such that $J = (m) \equiv \{km\colon k \in \mathbf{Z}\}$? The classical answer to this question is "Yes: for either J is $\{0\}$ or else we can take m to be the smallest positive integer in J". However, suppose we take the word "find" literally in the above question: is there an algorithm which, applied to any ideal J of $\mathbf{Z}$, will compute a nonnegative integer m such that $J = (m)$?

Consider the application of such an algorithm, if it exists, to the ideal

$$J \equiv (2) + \{ka_n\colon k \in \mathbf{Z}, n \geq 1\},$$

where $(a_n)_{n=1}^{\infty}$ is a binary sequence (that is, a sequence in $\{0, 1\}$). If m is the single generator of J that is produced by the algorithm, then it is clear that either $m = \pm 2$ or $m = \pm 1$. In the first instance, we have $a_n = 0$ for all n; in the second, choosing integers i, j, N such that $1 = 2i + ja_N$, we see that ja_N cannot equal 0, so $a_N = 1$. As the binary sequence $(a_n)_{n=1}^{\infty}$ is arbitrary, it follows that an algorithm for constructing single generators of ideals in **Z**—even one applicable only to nontrivial ideals of **Z**—can be transformed into an algorithm which, applied to any binary sequence (a_n), either proves that $a_n = 0$ for all n, or else computes a value N with $a_N = 1$. Let us call an algorithm of the latter type a *binary sequence decision algorithm* (bsda). It should require little thought to convince oneself that the existence of a bsda is highly unlikely; in fact, if one insists that all algorithms are recursive (the Church-Markov-Turing thesis), then the existence of a bsda is false, as it leads to a solution of the Halting Problem of recursive function theory: see [**Bridges-Richman**, Chapter 3, (1.3)].

Constructive mathematics, and in particular constructive algebra in the sense of the book under review, discusses questions, such as the one with which we started above, under a strict interpretation of "there exists" as "there is an algorithm that constructs". Following the approach begun by the late Errett Bishop, the authors prefer to take the notion of algorithm as primitive, rather than restrict themselves to any formal notion such as that provided by recursive function theory; compared with the recursive constructive mathematics advocated by Markov and Shanin [**Kushner**], Bishop's mathematics (which we shall refer to as BISH) has the advantage of clarity of expression—it looks like the mathematics which we were taught as undergraduates; at the same time, by not specifying a formal notion of algorithm, Bishop's approach cannot accommodate many of the counterexamples that are produced by recursive mathematics.

To answer our original question within BISH, we first define a subset S of a set X to be *detachable* if for each x in X either $x \in S$ or $x \notin S$; that is, if there is an algorithm which, applied to any element x of X, outputs 1 if $x \in S$, and 0 if $x \notin S$. It is easy to prove within BISH that a nonzero ideal J of **Z** is principal if and only if it is detachable; or, to be more precise, that

(i) there is an algorithm which, applied to any detachable nonzero ideal J of **Z**, computes a single generator of J; and

(ii) there is an algorithm which, applied to any principal ideal J of **Z** and any integer n, determines whether or not n belongs to J.

Note that when we say that an ideal J is nonzero, we mean that we can compute a positive integer that belongs to J. The reader might like to consider why the word "nonzero" cannot be deleted from statement (i).

The rather trivial mathematics associated with our question about ideals of **Z** serves to illustrate the basic difference between the constructive and classical approaches to mathematics. To take a less trivial, and from a constructive point of view much more challenging, problem, let us recall that, classically, a ring R is said to be (left) *Noetherian* if either of the following conditions obtains:

(N1) for each ascending chain $I_1 \subset I_2 \subset \cdots$ of left ideals of R there exists n such that $I_m = I_n$ for all $m \geq n$;

(N2) each ideal J of R is *finitely generated*, in the sense that there exist elements $x_1, \ldots, x_n$ of J such that each element of J can be written in the form $\sum_{i=1}^{n} r_i x_i$ for some $r_1, \ldots, r_n$ in R [**Cohn**, 2.2, Proposition 6].

Now consider the *classical Hilbert basis theorem*:

If R is a Noetherian ring, then so is the ring $R[X]$ of polynomials in the indeterminate X over R.

The standard classical proof of this theorem, as found in [**Cohn**, Chapter 11, Theorem 2], is constructive. But, constructively, even the simplest types of ring fail to satisfy the classical Noetherian hypotheses; consider, for example, the ring **Z**, a binary sequence (a_n), and the ascending chain (I_n) of ideals in **Z** where for each n, I_n is the ideal generated by $\{a_1, \ldots, a_n\}$: if there exists n such that $I_m = I_n$ for all $m \geq n$, then

either there exists $k \leq n$ such that $a_k = 1$,
or $a_k = 0$ for all $k \leq n$, in which case $a_m = 0$ for all $m \geq n$, and therefore for all m.

In other words, if, constructively, **Z** satisfies the classical Noetherian condition, then we can produce a bsda.

When faced with such a situation, in which an important classical property P of, for example, **Z** fails to hold constructively, the constructive mathematician looks for something better: a property P' which is classically equivalent to P and which does apply constructively to **Z**. In the case where P is the classical Noetherian property, the following definition introduces an appropriate constructive property P' which holds in the case $R = \mathbf{Z}$: a ring R is said to be (constructively) *Noetherian* if for each ascending chain $I_1 \subset I_2 \subset \cdots$ of finitely generated ideals of R there exists n such that $I_{n+1} = I_n$.

To see that this definition of Noetherian is classically equivalent to the classical one, suppose the ring R is not classically Noetherian. Then there exists an ideal J of R that is not finitely generated; so J contains a sequence $(x_n)_{n=1}^{\infty}$ of elements such that for each n, x_{n+1} is not in the ideal I_n generated by $\{x_1, \ldots, x_n\}$. Then $I_1 \subset I_2 \subset \cdots$ is an ascending chain of finitely generated ideal of R that fails to satisfy the constructive Noetherian condition.

Even with an appropriate constructive definition of Noetherian, we need something extra to prove a significant constructive version of the Hilbert basis Theorem. The authors describe two different ways of getting that something extra; in fact, they prove two quite different constructive theorems, each of which is classically,equivalent to the classical Hilbert basis theorem.

To discuss the first version in greater detail, we define a module M over a ring R to be *finitely presented* if, for some positive integer n, there exists a homomorphism φ from R^n onto M such that the kernel of φ is finitely generated. A ring R is said to be *coherent* if every finitely generated left ideal of R is finitely presented as an R-module. Classically, every Noetherian ring is coherent: this is a simple consequence of condition N2 above; but, by considering a binary sequence (a_n) and the ring $R \equiv \mathbf{Z}/I$, where I is the ideal of $\mathbf{Z}$ generated by $\{a_n n!: n \geq 1\}$, we can show that if, constructively, every Noetherian ring is coherent, then there exists a bsda.

Coherence is an extremely powerful constructive property: for example, if R is coherent, then we can construct a finite set of generators for the intersection of two finitely generated ideals of R; we can also construct a finite set of generators for the left annihilator $\{r \in R: rx = 0\}$ of an element x of R.

With an eye on the standard classical proof of the Hilbert basis theorem, for a coherent Noetherian ring R the authors prove first that $R[X]$ is coherent, and then that $R[X]$ is Noetherian. Since, classically, "Noetherian implies coherent," their results about $R[X]$ are classically equivalent to the Hilbert basis theorem in its usual form.

The authors' second attack on the Hilbert basis theorem uses weapons devised by J. Tennenbaum for his PhD thesis under Bishop's supervision. A *Noetherian basis function* for a ring R is a function φ with the following properties: the domain of φ is the set of all nonvoid finite sequences of elements of R; for $n = 2, 3, \ldots$, φ maps R^n into R^{n-1}; and if $(x_1, x_2, \ldots)$ is an infinite sequence of elements of R, then there exist infinitely many n such that $x_n = r_1x_1 + \cdots + r_{n-1}x_{n-1}$, where $(r_1, \ldots, r_{n-1}) = \varphi(x_1, \ldots, x_n)$. For example, $\mathbf{Z}$ has a Noetherian basis function. To see this, first define $\varphi(x) \equiv x$ for each x in $\mathbf{Z}$. Then consider integers $x_1, \ldots, x_n$, where $n \geq 2$. If $d \equiv \gcd(x_1, \ldots, x_{n-1}) = 0$, set $\varphi(x_1, \ldots, x_n) \equiv x_n$. If $d \neq 0$, choose integers $r_1, \ldots, r_{n-1}$ such that $\sum_{i=1}^{n-1} r_i x_i = x_n - r$, where r is the remainder on dividing x_n by d; then set $\varphi(x_1, \ldots, x_n) \equiv (r_1, \ldots, r_{n-1})$.

In their second version of the Hilbert basis theorem, the authors show that if the ring R has a Noetherian basis function, then so does $R[X]$. That this is a version of the Hilbert basis theorem follows from the fact that if a ring has Noetherian basis function and (algorithmically) decidable equality relation, then it is Noetherian.

To produce evidence that the authors' two versions are distinct, we need only take a binary sequence (a_n) and consider the ring $\mathbf{Z}/I$, where I is the ideal of $\mathbf{Z}$ generated by $\{a_n n!: n \geq 1\}$: as we observed above, it is unlikely that we will find an algorithm for proving that such rings are coherent; but, on the other hand, $\mathbf{Z}/I$ has a Noetherian basis function, and Tennenbaum's version of the Hilbert basis theorem can be used to show that $(\mathbf{Z}/I)[X]$ is Noetherian. Thus we have an example of a not uncommon phenomenon in constructive mathematics: a classical theorem which gives rise to several constructive versions which are classically equivalent but constructively distinct.

In his 1973 Colloquium Lectures for the American Mathematical Society, Errett Bishop commented on the lack of progress in constructive algebra, compared with that in constructive analysis (as developed in his seminal book, *Foundations of constructive analysis*, which largely consists of research carried out by him in the mid 1960s):

> "In spite of the pioneering efforts of Kronecker, and continued work by many algebraists, resulting in many deep theorems, the systematic constructivization of algebra would seem hardly to have begun. The problems are formidable. A very tentative suggestion is that we should restrict our attentions to algebraic structures endowed with some sort of topology, with respect to which all operations and maps are continuous. The work of Tennenbaum ... might provide some ideas of how to accomplish this. The task is complicated by the circumstance that no completely suitable constructive framework for general topology has yet been found."

(It is interesting to note that Mines and Richman were just beginning their research programme in constructive algebra around the time that Bishop gave his Colloquium Lectures. Ruitenburg joined the project several years later.) Although Bishop's concluding remark about topology remains true today, the book under review is a prime witness to the fact that, in constructive terms, algebra is no longer in its infancy compared with analysis. The book covers topics in algebra ranging from elementary results about the standard abstract algebraic structures to Galois theory, abelian groups, valuations, and Dedekind domains. Most of the material in the book is the product of the authors' own research over the last fifteen years. That product is a formidable achievement on their part.

A legitimate question to ask is, "Who, apart from a dedicated constructivist, might be interested in the book under review?" Obviously, logicians and workers in the foundations of mathematics, topos theorists, and theoretical computer scientists—particularly those interested in automated theorem proving **[Constable, Martin-Löf]**—will be drawn to it. Workers in recursive algebra, as practised by Nerode and others in the USA, will find it an invaluable reference. Algebraists looking for an excellent text for a graduate course with a difference would be well advised to consider adopting it. More generally, all who are concerned with algorithmic aspects of mathematics, or are interested in the computational origins of a branch of mathematics that can be made to appear abstraction *par excellence*, will appreciate the insight and technical flair displayed by its authors.

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Holomorphic functions and integral representations in several complex variables, by R. Michael Range. Graduate Texts in Mathematics, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1986, xi + 386 pp., $49.50. ISBN 0-387-96259-x

Complex function theory of one variable can be developed on the basis of three different approaches.

(a) The (so-called) *Weierstrass* approach, namely the fact that holomorphic functions can be locally represented by their Taylor expansions. Here the basic properties of the ring $\mathscr{O}^{(1)}$ of convergent power series in one variable such as $\mathscr{O}^{(1)}$ being a principal ideal ring become important;

(b) The (so-called) *Riemann* approach, based on the fact that holomorphic functions can be characterized as those differentiable functions $f = g + ih$ in $z = x + iy$ satisfying the Cauchy-Riemann equations:

$$(1) \qquad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0.$$

Here the "good" properties of the system (1) of partial differential equations are the essential feature. Namely, it is elliptic, linear with constant coefficients and intimately related to the Laplace operator $\Delta = 4\partial^2/\partial z\partial\bar{z}$ with all its wonderful, well-known properties. Furthermore, (1) has as its natural geometric interpretation the conformality of biholomorphic maps.

(c) The (so-called) *Cauchy* approach, based on the Cauchy integral formula for holomorphic functions. The properties of the integral operator(s) with the Cauchy kernel are used as the most powerful tools from this point of view. In particular one has the formula

$$(2) \qquad f(z) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_{\Omega}\frac{\partial f/\partial\bar{\zeta}}{\zeta - z}\,d\zeta\wedge d\bar{\zeta}$$

which holds for all domains $\Omega \Subset \mathbf{C}$ the boundary of which consists of a finite number of disjoint C^1 Jordan curves, for all $f \in C^1(\overline{\Omega})$ and for all $z \in \Omega$. It can be used very successfully in this approach.

(It should be pointed out that the association of the names of Weierstrass, Riemann and Cauchy with these approaches can be justified only partially from the historical viewpoint. For some interesting details about this see for instance **[27]**.)

Most presentations of basic function theory use a pragmatic mixture of the approaches (a)–(c). It can, however, also be quite interesting to