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Global analysis on foliated spaces by Calvin C. Moore and Claude Schochet. Mathematical Sciences Research Institute Publications, vol. 9, Springer-Verlag, New York, Berlin, Heidelberg, 1988, 377 pp., $34.00. ISBN 0-387-96664-1

Let M be a closed compact oriented Riemannian surface. The Euler number of M, $\chi(M)$, is given by $\sum(-1)^i \dim H^i(M;R)$, where $H^i(M;R)$ is the ith de Rham cohomology group of M. This integer determines M up to diffeomorphism. The Gauss-Bonnet theorem states that

$$\chi(M) = \int_M (1/2\pi)\Omega$$

where Ω is the curvature of the Levi-Civita connection on the tangent bundle of M. Now $d[(1/2\pi)\Omega] = 0$ so it defines a cohomology class on M, its Euler class, and we may interpret the theorem as saying that the Euler number of M (an analytic invariant) is given by the integral over M of a certain characteristic cohomology class (a topological invariant).

This simple theorem is at once the genesis and a paradigm for the index theory of elliptic operators, a theory which relates topological invariants of differential structures on the one hand to analytical invariants on the other. The central theorem in this theory is the Atiyah-Singer index theorem [AS]. Briefly, it says the following. Let M be a closed compact oriented manifold, let $E = (E_0, E_1, \dots, E_k)$ be a family of complex vector bundles over M, and let $d = (d_0, d_1, \dots, d_{k-1})$ be a family of differential operators, d_i mapping sections of E_i to sections of E_{i+1}. Suppose that $d_i \cdot d_{i-1} = 0$ and that the differential complex (E, d) satisfies a technical condition called ellipticity. Roughly speaking, ellipticity means that the associated Laplacians (see below) differentiate in all possible directions. Ellipticity implies, among other things, that for all i, $H^i(E, d) = \ker d_i / \operatorname{image} d_{i-1}$ is a finite dimensional vector space. Define the index of (E, d) to be

$$I(E, d) = \sum_{i=0}^{k} (-1)^i \dim H^i(E, d).$$

The Atiyah-Singer theorem relates this integer invariant to the integral over M of certain characteristic cohomology classes. Specifically, it states that

$$I(E,d) = \int_M \psi^{-1}[\mathrm{ch}(E)] \cdot \mathrm{td}(M)$$

where $\mathrm{td}(M)$ is the Todd class of TM, the tangent bundle of M, (a characteristic de Rham cohomology class on M), $\mathrm{ch}(E)$ is the Chern character of a virtual bundle on TM constructed from (E,d), (a characteristic de Rham class on TM), and $\psi\colon H^*(M;R) \to H^*(TM;R)$ is the Thom isomorphism.

This theorem encompasses the Gauss-Bonnet theorem, the Riemann-Roch theorem, the spinor index theorem, and the signature theorem. Its extension to G invariant operators (G a compact Lie group) is called the G index theorem and it generalizes the Lefschetz fixed point theorem to general elliptic complexes. The index theorem and its generalizations have been enormously useful in topology, geometry, physics and representation theory.

As befits a theorem of such depth and generality, the index theorem has several proofs using different techniques. The original proof **[P]** was modeled on Hirzebruch's proof of the Riemann-Roch theorem. This proof used cobordism and did not lend itself to certain natural generalizations where the appropriate cobordism groups were not known. The proof in **[AS]** was modeled on Grothendieck's proof of Riemann-Roch and used K-theory in place of cobordism. There is a third proof based on the heat equation method, **[ABP, G1]**, and this proof has led to further generalizations of the Atiyah-Singer theorem, most notably to index theorems on open manifolds **[R]**, to local index theorems, references below, to index theory for operators defined along the leaves of a foliated manifold (the work of A. Connes), and to Lefschetz theory on foliated manifolds **[HL]**.

The heat equation proof of the classical index theorem proceeds as follows. On each E_i choose an Hermitian metric and use these metrics to construct the adjoints d_i^* of the d_i. The ith Laplacian $\Delta_i = d_i^* d_i + d_{i-1} d_{i-1}^*$ of (E,d) is then an operator acting on smooth sections of E_i, and it extends to a densely defined unbounded operator on $L^2(E_i)$, the L^2 sections of E_i. In addition, it is selfadjoint, nonnegative and has spectrum $0 = \lambda_0 < \lambda_1 < \cdots$. Ellipticity implies that the eigenspace $E(i,j)$ of $L^2(E_i)$ corresponding to λ_j is finite dimensional. As $d_i \cdot \Delta_i = \Delta_{i+1} \cdot d_i$, $d_i(E(i,j)) \subset E(i+1,j)$, and for $j > 0$, it is easy to check that the sequence

$$(**) \qquad \cdots \to E(i,j) \xrightarrow{d_i} E(i+1,j) \to \cdots$$

is exact. For $j = 0$, we have $E(i,0) = \ker \Delta_i$ which by Hodge theory is isomorphic to $H^i(E,d)$. Now $L^2(E_i) = \bigoplus_j E(i,j)$, and if we define $\exp(-t\Delta_i)\colon L^2(E_i) \to L^2(E_i)$ by $\exp(-t\Delta_i)|E(i,j)$ is multiplication by $\exp(-t\lambda_j)$, we obtain an operator of trace class, i.e.

$$\mathrm{tr}[\exp(-t\Delta_i)] = \sum_{j=0}^{\infty} \exp(-t\lambda_j) \dim E(i,j) < \infty.$$

Note that $\lim_{t\to\infty} \operatorname{tr}[\exp(-t\Delta_i)] = \dim E(i,0) = \dim H^i(E,d)$. Thus

$$\lim_{t\to\infty} \sum_{i=0}^{k} (-1)^i \operatorname{tr}[\exp(-t\Delta_i)] = I(E,d).$$

In fact, $\sum_{i=0}^{k} (-1)^i \operatorname{tr}[\exp(-t\Delta_i)]$ is independent of t. To see this, note that the exact sequences $(**)$ imply that for fixed t, and $j > 0$,

$$\sum_{i=0}^{k} (-1)^i \exp(-t\lambda_j) \dim E(i,j) = 0.$$

Thus

$$\sum_{i=0}^{k} \sum_{j=1}^{\infty} (-1)^i \exp(-t\lambda_j) \dim E(i,j) = 0,$$

and so we have

$$\begin{aligned}\sum_{i=0}^{k} (-1)^i \operatorname{tr}[\exp(-t\Delta_i)] &= \sum_{i=0}^{k} \sum_{j=0}^{\infty} (-1)^i \exp(-t\lambda_j) \dim E(i,j) \\ &= \sum_{i=0}^{k} (-1)^i \dim E(i,0) = I(E,d).\end{aligned}$$

The index theorem is then established by considering the behavior of $\operatorname{tr}[\exp(-t\Delta_i)]$ as $t \to 0$. The operator $\exp(-t\Delta_i)$ is a smoothing operator on $L^2(E_i)$ so it has a smooth Schwartz kernel $K_{i,t}(x,y)$. That is, $K_{i,t}(x,y)$ is a smooth section of the bundle $\operatorname{Hom}(E_i, E_i)$ over $M \times M$ and for any section s of E_i,

$$[\exp(-t\Delta_i)s](x) = \int_M K_{i,t}(x,y)s(y)\,dy,$$

where dy is the volume form of the metric on M. $\operatorname{Tr}[\exp(-t\Delta_i)]$ is then given by $\int_M [\operatorname{tr} K_{i,t}(y,y)]\,dy$. (Note that the particular section of $\operatorname{Hom}(E_i, E_i)$ which represents $\exp(-t\Delta_i)$ depends on the metric on M, but the integral $\int_M [\operatorname{tr} K_{i,t}(y,y)]\,dy$ does not.)

Now as $t \to 0$, $\exp(-t\Delta_i) \to \mathrm{Id}$ and $K_{i,t}(x,y)$ is converging to $\delta_{i,x}(y)$, a Dirac section of $\operatorname{Hom}(E_i, E_i)$. Thus for t small, $\exp(-t\Delta_i)$ is essentially a *local* operator, that is, modulo a rapidly decreasing (in t) error, $[\exp(-t\Delta_i)s](x)$ depends only on s is a neighborhood of x. It is thus reasonable to expect that for t small, $K_{i,t}(x,y)$ can be approximated reasonably well by a kernel constructed out of local data and that as $t \to 0$, this approximation gets better and better. This is in fact true and leads to an asymptotic expansion for $\operatorname{tr}[K_{i,t}(y,y)]$ whose coefficients are locally computable as functions of the coefficients of the Δ_i and the metrics on M and the bundles E_i, **[S, MP]**. As

$$I(E,d) = \int_M \sum_{i=0}^{k} (-1)^i \operatorname{tr}[K_{i,t}(y,y)]\,dy$$

and the left-hand side is independent of t, the alternating sum of the zeroth order terms in the asymptotic expansions must also yield $I(E,d)$ when integrated over M. It was Gilkey [G2] and Patodi [Pt] who showed, using invariant theory, that for the classical complexes this locally constructed differential form represents the de Rham cohomology class given in the Atiyah-Singer recipe. The proof is then finished by showing that in a suitable sense, namely K-theory, the classical elliptic complexes generate all the elliptic complexes. (Recently, Getzler [Gz], Bismut [B], Berline and Vergne [BV], and Donnelly [D] have shown how to prove directly that for the classical operators, the integrand in question represents the Atiyah-Singer class.)

The book of Moore and Schochet is an exposition of the work of Alain Connes on the extension of index theory to operators defined along the leaves of a foliated manifold. Connes' proof of course uses the heat equation method.

Let F be a p dimensional foliation of an m dimensional compact Riemannian manifold M and, for simplicity, assume that M and F are oriented. F is a partition of M into p dimensional submanifolds (called the leaves of F) which locally looks like a family of parallel p planes in R^m. Globally the behavior of the leaves can be quite complicated and in general they are not compact submanifolds. As above, let (E, d) be a differential complex over M. The d_i are now required to differentiate only in directions tangent to the leaves and the restriction of the complex to each leaf is required to be elliptic. Let L be a leaf of F and $\Delta_{i,L}$ the ith Laplacian of the complex restricted to L. As L is not necessarily compact, we cannot conclude that $\ker(\Delta_{i,L})$ is finite dimensional, nor that $\Delta_{i,L}$ has discrete spectrum. However, it is still possible to prove an index theorem using the heat equation method. The operator $\exp(-t\Delta_{i,L})$ is still defined (by the spectral mapping theorem) and it has a smooth Schwartz kernel, denoted $K_{i,t,L}(x, y)$, with respect to the volume form dy induced on L by the metric on M. As $t \to 0$, $\operatorname{tr}[K_{i,t,L}(y, y)]$ still has an asymptotic expansion whose coefficients are locally computable and are the same as in the classical case. At $t \to \infty$, $\exp(-t\Delta_{i,L})$ converges to $P_{i,L}$ the projection onto $\ker(\Delta_{i,L})$ and $P_{i,L}$ has a smooth Schwartz kernel, denoted $P_{i,L}(x, y)$.

To state Connes' index theorem we now need some way to amalgamate the information on individual leaves. This is where invariant transverse measures enter the picture. Simply put, a transverse measure du on M assigns a measure to each $m - p$ dimensional submanifold of M which is transverse to the leaves of F. To say that the transverse measure is invariant means that the measure assigned to a transverse submanifold remains unchanged if the submanifold is deformed by a shear along the leaves of F.

Now each leaf L has the volume form dy induced by the metric on M. This leafwise measure combined with the transverse measure produces a measure $dy\,du$ on M. Define the Connes' index of (E, d) by

$$I_C(E, d) = \int_M \sum_{i=0}^{k} (-1)^i \operatorname{tr}[P_{i,L}(y, y)]\, dy\, du.$$

The Connes' index theorem then states that

$$I_C(E, d) = \int_M \psi^{-1}[\mathrm{ch}(E)] \cdot \mathrm{td}(F)\, du$$

where $\mathrm{td}(F)$ is the Todd class of the tangent bundle TF of F, $\mathrm{ch}(E)$ is the Chern character of a virtual bundle over TF constructed out of (E, d), and ψ: $H^*(M, R) \to H^*(TF, R)$ is the Thom isomorphism. To evaluate this integral, let w be a differential form on M representing $\psi^{-1}[\mathrm{ch}(E)] \cdot \mathrm{td}(F)$, and let w_p be its p dimensional part. As the leaves of F are oriented, we may view w_p as a measure on each leaf. Combine this measure with du to obtain a measure on M. Integrate this measure.

In outline, the proof of the theorem is essentially the same as in the classical case. One shows that

$$\int_M \sum_{i=0}^{k} (-1)^i \,\mathrm{tr}[K_{i,t,L}(y, y)]\, dy\, du,$$

is independent of t, that its limit as $t \to \infty$ is $I_C(E, d)$, and its limit as $t \to 0$ is $\int_M \psi^{-1}[\mathrm{ch}(E)] \cdot \mathrm{td}(F)\, du$.

Of course, there are a number of rather deep technical difficulties to be overcome. The quest for the proof leads through functional analysis, C^* and von Neumann algebras, topological groupoids, characteristic classes and K-theory along a foliation, and the theory of pseudodifferential operators. It is a long but very rewarding journey and Moore and Schochet have performed a valuable service in putting all this material in one place in an easily readable form. As their approach is quite general (some may feel that it is too general in spots) the book contains a wealth of information. It is not for those who wish an overview of the index theorem on foliated manifolds. However, for those wishing a comprehensive proof of Alain Connes' beautiful theorem, this book is indispensible. (For those who are already familiar with the material necessary to understand the statement of Connes' theorem, **[R2]** provides a short proof.)

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Elliptic functions and rings of integers, by Ph. Cassou-Noguès and M. J. Taylor. Progress in Mathematics, vol. 66, Birkhäuser, Boston, Basel and Stuttgart, 1987, xvi + 198 pp., $29.50. ISBN 0-8176-3350-2

The idea of constructing prescribed algebraic number fields by means of the values of real or complex functions is usually referred to as the 'Jugendtraum,' a word used by Kronecker in an 1880 letter to Dedekind [**8**]. Hilbert made the realization of this idea the twelfth problem of his 1900 address [7], adding that he considered "...this problem as one of the most profound and far-reaching in the theory of numbers and of functions." In the present book, Cassou-Noguès and Taylor give an exposition of Taylor's recent work on an "integral Jugendtraum" for the rings of integers of ray class fields of imaginary quadratic fields. The object of this work is to construct by means of elliptic functions explicit generators of such rings of integers either as algebras or as Galois modules.

The two examples which motivated the Jugendtraum were provided by the finite abelian extensions of either the rational numbers $\mathbf{Q}$ or of an imaginary quadratic field. By the Kronecker-Weber Theorem, every abelian extension of $\mathbf{Q}$ is contained in a cyclotomic field of the form $\mathbf{Q}(\zeta_n)$, where $\zeta_n = \exp(2\pi i/n)$ is a primitive nth root of unity for some positive integer n. Thus the values of the function $e(z) = \exp(2\pi i z)$ at rational z generate all abelian extensions of $\mathbf{Q}$. One can view these z as the points of finite order on the one-dimensional real torus $\mathbf{R}/\mathbf{Z}$. Suppose now that $\mathbf{Q}$ is replaced by an imaginary quadratic field K. In this case the torus $\mathbf{R}/\mathbf{Z}$ may be replaced by a two-dimensional torus $\mathbf{C}/\Omega$, where Ω is a nonzero ideal of $\mathscr{O}_K$ and we view K as a fixed subfield of $\mathbf{C}$. An elliptic function for $\mathbf{C}/\Omega$ is a meromorphic function of $z \in \mathbf{C}$ whose value at z depends only on $z \bmod \Omega$. A division point of $\mathbf{C}/\Omega$ is a point of finite order on $\mathbf{C}/\Omega$. By work of Weber, Feuter and Hasse, there are elliptic functions for $\mathbf{C}/\Omega$ such that every abelian extension of K is contained in a ray class field generated over K by the values of these functions at suitable division points. There are different combinations of elliptic functions which generate the