REFERENCES

1. T. Jech, *Set theory*, Academic Press, New York, 1978.

JAMES E. BAUMGARTNER
DARTMOUTH COLLEGE

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 1, January 1989

0273-0979/89 $1.00 + $.25 per page

Dimensions of ring theory, by Constantin Nastasescu and Freddy van Oystaeyen. D. Reidel Publishing Company, Dordrecht, Boston, Lancaster and Tokyo, 1987, xi + 360 pp., $74.00. ISBN 90-277-2461-x

While mathematics is certainly not a "science of measurement", mathematicians do seek to "take the measure" of everything they study, often in the form of numerical, cardinal, or ordinal invariants—for example, to measure the deviation of a certain system from some ideal situation, to measure how likely or unlikely a certain object is to enjoy a certain property, or simply to measure the progress in some inductive procedure. Many such invariants have evolved, either directly or through analogy, from the Euclidean dimensions with which we measure our "real" world, and thus many invariants are called "dimensions" of some sort, usually decorated with one or more adjectives. Ring theory has its share of such dimensions, attached to both rings and modules, and since these dimensions have evolved in an algebraic rather than a geometric environment, their connection with Euclidean dimension may not be readily apparent. To illustrate, we discuss three examples—Goldie dimension, Krull dimension, and Gelfand-Kirillov dimension.

Goldie dimension. Vector space dimension cannot be applied directly to arbitrary modules because most modules do not have bases, and even among those that do (namely the free modules), one can find modules in which different bases may have different cardinalities. The dimension of a vector space V can, however, be expressed as the number of terms in a decomposition of V into a direct sum of irreducible subspaces, or as the maximum number of terms occurring in decompositions of V into direct sums of nonzero subspaces. Since the complexity of a module need not be reflected by direct sum decompositions, one looks at decompositions of submodules along with decompositions of a given module. Thus the *Goldie dimension* (also called the *uniform dimension*, the *uniform rank*, or the *Goldie rank*) of a module M is defined to be the supremum of the number of nonzero terms in any direct sum decomposition of any submodule of M.

This dimension arose in Goldie's 1958 development of noncommutative rings of fractions [3], since one necessary condition for a ring R to have a simple artinian ring of fractions Q is that the Goldie dimension of R (considered as a module over itself) be finite. More specifically, such a Q must be isomorphic (by the Artin-Wedderburn Theorem) to the ring of $n \times n$ matrices over a division ring, and n is exactly the value of the Goldie dimension of R. Goldie dimension has in particular been found to be an important invariant for representations of a finite-dimensional (complex) semisimple Lie algebra $\mathfrak{g}$. The annihilators of the irreducible representations of $\mathfrak{g}$ are the *primitive* ideals P of the enveloping algebra $U(\mathfrak{g})$, and each of the factor rings $U(\mathfrak{g})/P$ has a simple artinian ring of fractions. The primitive ideals of $U(\mathfrak{g})$ can be parametrized by the dual $\mathfrak{h}^*$ of a Cartan subalgebra $\mathfrak{h}$, and Joseph [4] discovered that $\mathfrak{h}^*$ is a disjoint union of infinite subsets Λ_i such that the Goldie dimensions of the primitive factor rings of $U(\mathfrak{g})$ corresponding to $\lambda \in \Lambda_i$ are given by polynomial functions of λ.

Krull dimension. This evolved from the geometric dimension of algebraic varieties. One way to measure the dimension of an irreducible variety V is to count the number of proper inclusions in a chain

$$V \supset V_1 \supset V_2 \supset \cdots \supset V_n$$

of irreducible subvarieties contained in V. Such a chain corresponds to a chain

$$P \subset P_1 \subset P_2 \subset \cdots \subset P_n$$

of prime ideals in a polynomial ring $k[x_1, \ldots, x_d]$, where P is the ideal defining V (that is, V is the variety whose points are the common zeroes of the polynomials in P). In 1923, Noether [10] proved (over an infinite field k, but that hypothesis is not needed) that this gives the correct answer: the dimension of V equals the maximum number of proper inclusions in chains of prime ideals ascending from P. Chain-counting then became the standard ideal-theoretic means of defining "dimension" in any commutative ring R, namely as the supremum of the lengths of finite chains of prime ideals in R. This value is now known as the *Krull dimension* of R, honoring Krull's fundamental contribution [7] in developing this concept into a powerful tool for arbitrary (i.e., nonpolynomial) commutative noetherian rings.

Since noncommutative rings generally have a meager supply of prime ideals, a refinement of the classical Krull dimension was introduced by Rentschler and Gabriel [12] and extended to arbitrary ordinal values by Krause [5]. One may motivate it with the following observation. If R is a commutative integral domain whose Krull dimension is a positive integer n, there is a nonzero element x in R such that the ring R/Rx has Krull dimension $n-1$. Then there is an infinite descending chain of ideals

$$R \supset Rx \supset Rx^2 \supset Rx^3 \supset \cdots$$

such that each of the factor modules Rx^i/Rx^{i+1} is isomorphic to R/Rx and thus is "as large as" a ring of Krull dimension $n-1$. Hence, one is led to define a Krull dimension on modules, and to do so by considering the successive factor modules in descending chains of submodules; the concept is then applied to a ring R by considering R as a module over itself.

First, artinian modules are assigned Krull dimension 0. If α is an ordinal and M is a module which does not already have a Krull dimension less than α, then M is assigned Krull dimension α if and only if for each descending chain $M_1 \supseteq M_2 \supseteq \cdots$ of submodules of M, all but finitely many of the factors M_i/M_{i+1} have been assigned Krull dimensions less than α. While not all modules can be assigned a Krull dimension, all noetherian modules do receive a Krull dimension. This version of Krull dimension is known to agree with the classical version for commutative noetherian rings; even somewhat more generally (e.g., for noetherian rings satisfying a polynomial identity).

One example of the use of Krull dimension is to estimate the number of generators needed for a module, as in Stafford's extension of the Forster-Swan Theorem **[13]**. Namely, let M be a finitely generated left module over a noetherian ring R, and for each prime ideal P of R let Q_P be Goldie's simple artinian ring of fractions of R/P and let $g(M, P)$ be the minimal number of generators for $Q_P \otimes_R M$ as a Q_P-module. Then the minimal number of generators for M is bounded by the supremum of the numbers

$$g(M, P) + (\text{Krull dimension of } R/P)$$

as P ranges over the prime ideals of R. (This supremum actually bounds the *stable* number of generators for M, and so in the case $M = R$ it provides an upper bound for the K-theoretic *stable range* of R.)

Gelfand-Kirillov dimension. Returning to geometry, a rather loose way to think of the dimension of an algebraic variety V is as the "minimum number of independent variables" needed to define algebraic functions on V. More precisely, the dimension of V equals the transcendence degree (over the base field) of the coordinate ring R of V. The transcendence degree of R can be thought of, in turn, as a growth rate for R relative to a set of generators. This is easiest to see in the case of a polynomial ring $k[x_1, \ldots, x_d]$ over a field k—the number of monomials of degree at most n in the variables $x_1, \ldots, x_d$ is a polynomial function of n, of degree d. Gelfand and Kirillov introduced a dimension along these lines in 1966, defined for a finitely generated algebra A over a field **[1, 2]**. Choose a finite-dimensional subspace W which generates A as an algebra, and for positive integers n let W_n be the subspace of A spanned by all products of at most n elements from W. The *Gelfand-Kirillov dimension* of A is then defined by the formula

$$\text{GK.dim}(A) = \limsup \frac{\log \dim W_n}{\log n}.$$

(It is not hard to show that this number is independent of the choice of W.) For instance, A has Gelfand-Kirillov dimension 0 precisely if A is finite-dimensional. For commutative finitely generated algebras, the Gelfand-Kirillov dimension agrees with the Krull dimension, and is therefore an integer. In general, however, every real number greater than 2 can appear as the Gelfand-Kirillov dimension of some finitely generated algebra.

This dimension is an important tool in the representation theory of a Lie algebra $\mathfrak{g}$; in particular, it is more tractable and more easily computable than the Krull dimension in the enveloping algebra $U(\mathfrak{g})$. (For instance, the Gelfand-Kirillov dimension is independent of any left- versus right-handed considerations, whereas in the nonsolvable case it is an open question whether factor rings $U(\mathfrak{g})/I$ have the same Krull dimension as left and right modules.) To make a parallel with geometry, we recall the fact that in the coordinate ring of an irreducible algebraic variety V, the maximal ideals all have the same height, which equals the dimension of V. (The *height* of a prime ideal P in a ring is the supremum of the lengths of chains of prime ideals descending from P.) This is usually not true in an enveloping algebra $U(\mathfrak{g})$. However, when $\mathfrak{g}$ is solvable, Tauvel **[14]** has proved that the Gelfand-Kirillov dimension exactly makes up the difference:

$$\dim(\mathfrak{g}) - \text{height}(M) = \text{GK.dim}(U(\mathfrak{g})/M)$$

for all maximal ideals M in $U(\mathfrak{g})$.

Relationships. Since various dimensions measure different aspects of a ring or a module, there is considerable interest in relationships that might exist among these dimensions. First, it should be mentioned that Goldie dimension is essentially completely independent of other dimensions—for instance, thinking of the lattice of submodules of a module, the Goldie dimension measures the "horizontal" size of this lattice, while the Krull dimension measures its "vertical" size. Second, the Gelfand-Kirillov dimension of an algebra A is "usually" greater than or equal to the Krull dimension (for instance, when A is a factor algebra of an enveloping algebra), and is often strictly greater. For example, if A is the nth Weyl algebra over a field of characteristic zero (generated by elements $x_1, \dots, x_n, y_1, \dots, y_n$ such that $x_i x_j = x_j x_i$ and $y_i y_j = y_j y_i$ while $x_i y_j - y_j x_i = \delta_{ij}$ for all i, j), then A has Krull dimension n and Gelfand-Kirillov dimension $2n$. Finally, we mention the global (homological) dimension (whose definition occurs in all homological algebra texts). It is well known that for a commutative noetherian ring, if the global dimension is finite then it equals the Krull dimension. For noncommutative noetherian rings, there is a long-standing conjecture that if the global dimension is finite then it is greater than or equal to the Krull dimension, which has been proved in certain cases (e.g., for noetherian rings satisfying a polynomial identity).

Dimensions of ring theory by Nastasescu and van Oystaeyen aims to be what would once have been called a "primer" of dimension theory. Essentially from the ground up, it develops Goldie dimension, Krull dimension, Gabriel dimension (a categorically-defined variant of Krull dimension), homological dimension, and Gelfand-Kirillov dimension for general rings, modules, and algebras, including some computations of these dimensions in important cases, and some illustrative applications. Since the Goldie, Krull, and Gabriel dimensions for a module can be defined entirely in terms of the lattice of submodules, the authors develop these dimensions first in the context of modular lattices and then specialize to modules. This in turn entails the development of a small amount of lattice theory, the payoff for which is that the ensuing discussions using the lattice-theoretic language then emphasize the similarities among these dimensions. (While the lattice-theoretic formulations of Goldie and Krull dimensions are well known, the reformulation of Gabriel dimension in lattice-theoretic terms by Lanski **[8]** is not, and the present book provides the first expository development of it.) In order to have some interesting classes of rings at hand, several sections are devoted to developing normalizing extensions, group-graded rings, and fixed rings. The book also includes over 200 exercises, and bibliographical comments to each chapter.

The authors' point of view is that treating all these dimensions together provides a unifying approach to ring theory. However, a student just starting out in the area is likely to find the presentation here to be a forbidding forest of technical detail. In particular, the authors' straight-line axiomatic development (definition, lemma, theorem, proof,...) includes hardly any discussion of the motivations for considering these dimensions. Experts, on the other hand, will already have their favorite sources for these concepts (excepting perhaps the lattice-theoretic treatment of Gabriel dimension), such as **[11]** for Goldie dimension, **[9]** for Krull dimension, **[6]** for Gelfand-Kirillov dimension, and any number of homological algebra texts for homological dimension.

Finally, the book provides a compendium of typographical errors. For instance, there are (in quantity) omitted, extra, or incorrect words (including incorrect mathematical terms) and inconsistent symbols (random variations in size, font, italics, capitals). Occasionally definitions or theorems are mis-stated, hypotheses or portions of proofs are omitted, or proofs are incorrect. Goldie dimension suffers particularly: the Goldie dimension of a module is never explicitly defined (the reader must assume—correctly—that it equals the lattice-theoretically defined Goldie dimension applied to the lattice of submodules), and in one section Goldie dimension is inexplicably transmuted into "coirreducible dimension."

In conclusion, I can only recommend this book to readers with a large supply of both patience and red ink.

REFERENCES

1. I. M. Gelfand and A. A. Kirillov, *On fields connected with the enveloping algebras of Lie algebras* (Russian), Dokl. Akad. Nauk SSSR **167** (1966), 503–505; [Translation: Soviet Math. Dokl. **7** (1966), 407–409].

2. ——, *Sur les corps liés aux algèbres enveloppantes des algèbres de Lie*, Publ. Math. I.H.E.S. **31** (1966), 5–19.

3. A. W. Goldie, *The structure of prime rings under ascending chain conditions*, Proc. London Math. Soc. (3) **8** (1958), 589–608.

4. A. Joseph, *Goldie rank in the enveloping algebra of a semisimple Lie algebra*. I, J. Algebra **65** (1980), 269–283.

5. G. Krause, *On the Krull-dimension of left noetherian left Matlis-rings*, Math. Z. **118** (1970), 207–214.

6. G. R. Krause and T. H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, Research Notes in Math., no. 116, Boston-London-Melbourne, Pitman Press, 1985.

7. W. Krull, *Primidealketten in allgemeinen Ringbereichen*, Sitzungsberichte Heidelberg. Akad. Wissenschaft (1928) 7. Abhandl., 3–14.

8. C. Lanski, *Gabriel dimension and rings with involution*, Houston J. Math. **4** (1978), 397–415.

9. J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, Wiley-Interscience, Chichester-New York-Brisbane-Toronto-Singapore, 1987.

10. E. Noether, *Eliminationstheorie und allgemeine Idealtheorie*, Math. Ann. **90** (1923), 229–261.

11. D. S. Passman, *The algebraic structure of group rings*, Wiley-Interscience, New York-London-Sydney, 1977.

12. R. Rentschler and P. Gabriel, *Sur la dimension des anneaux et ensembles ordonnés*, C. R. Acad. Sci. Paris Sér. A **265** (1967), 712–715.

13. J. T. Stafford, *Generating modules efficiently: algebraic K-theory for noncommutative Noetherian rings*, J. Algebra **69** (1981), 312–346.

14. P. Tauvel, *Sur les quotients premiers de l'algèbre enveloppante d'une algèbre de Lie résoluble*, Bull. Soc. Math. France **106** (1978), 177–205.

K. R. GOODEARL
UNIVERSITY OF UTAH

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 1, January 1989

0273-0979/89 $1.00 + $.25 per page

Global analysis on foliated spaces by Calvin C. Moore and Claude Schochet. Mathematical Sciences Research Institute Publications, vol. 9, Springer-Verlag, New York, Berlin, Heidelberg, 1988, 377 pp., $34.00. ISBN 0-387-96664-1

Let M be a closed compact oriented Riemannian surface. The Euler number of M, $\chi(M)$, is given by $\sum(-1)^i \dim H^i(M;R)$, where $H^i(M;R)$ is the ith de Rham cohomology group of M. This integer determines M up to diffeomorphism. The Gauss-Bonnet theorem states that

$$\chi(M) = \int_M (1/2\pi)\Omega$$

where Ω is the curvature of the Levi-Civita connection on the tangent bundle of M. Now $d[(1/2\pi)\Omega] = 0$ so it defines a cohomology class on M, its Euler class, and we may interpret the theorem as saying that the Euler number of M (an analytic invariant) is given by the integral over M of a certain characteristic cohomology class (a topological invariant).

This simple theorem is at once the genesis and a paradigm for the index theory of elliptic operators, a theory which relates topological invariants of differential structures on the one hand to analytical invariants on the other. The central theorem in this theory is the Atiyah-Singer index theorem [AS]. Briefly, it says the following. Let M be a closed compact oriented manifold, let $E = (E_0, E_1, \dots, E_k)$ be a family of complex vector bundles over M, and let $d = (d_0, d_1, \dots, d_{k-1})$ be a family of differential operators, d_i mapping sections of E_i to sections of E_{i+1}. Suppose that $d_i \cdot d_{i-1} = 0$ and that the differential complex (E, d) satisfies a technical condition called ellipticity. Roughly speaking, ellipticity means that the associated Laplacians (see below) differentiate in all possible directions. Ellipticity implies, among other things, that for all i, $H^i(E,d) = \ker d_i / \operatorname{image} d_{i-1}$ is a finite dimensional vector space. Define the index of (E, d) to be

$$I(E,d) = \sum_{i=0}^{k} (-1)^i \dim H^i(E,d).$$