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STANLEY BURRIS
UNIVERSITY OF WATERLOO

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 1, January 1989

0273-0979/89 $1.00 + $.25 per page

Generalized solutions of nonlinear partial differential equations, by Elemér E. Rosinger. Mathematical Studies, vol. 146, North-Holland, Elsevier Science Publishers, Amsterdam, 1987, xvii + 409 pp., US $ 92.00; Dfl 175.00. ISBN 0-444-70310-1

Why the need for generalized solutions of partial differential equations? It has been recognized that many equations of physics do not have classical solutions (for instance shock wave solutions of systems of conservation laws). Distribution solutions—usually called "weak solutions"—of the model equation

$$u_t + uu_x = 0$$

are defined as those integrable functions u which satisfy: $\forall \psi \in \mathscr{C}^\infty(\mathbf{R}^2)$ with compact support

$$(1) \qquad \iint \left[u(x,t)\frac{\partial}{\partial t}\psi(x,t) + \frac{1}{2}u^2(x,t)\frac{\partial}{\partial x}\psi(x,t)\right] dx\, dt = 0.$$

In the case of linear equations a detailed theory has been developed **[15, 7]**. However the situation is far from being satisfactory. Lewy **[8]** showed that the very simple linear equation

$$(2) \qquad \frac{\partial}{\partial x_1}u + i\frac{\partial}{\partial x_2}u - 2i(x_1 + ix_2)\frac{\partial}{\partial x_3}u = f$$

fails to have even local solutions for a large class of C^∞ second members $f(x_1, x_2, x_3)$. Concerning nonlinear equations the situation is by far worse, but several weak solution methods have been developed as surveyed in **[9]**.

The contention of the author of the book under review is that this situation stems from the lack of a suitable mathematical setting which could offer a convenient nonlinear concept of generalized solutions. This setting cannot be the one of distributions (or ultra distributions, hyperfunctions, ...) as is quite clear from an analysis of classical facts (Part I of the book under review and below). Now let us give an example from physics. Consider the following system of one dimensional elasticity **[4]**:

$$(3) \qquad \begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p - S)_x = 0, \\ (\rho e)_t + [\rho e u + (p - S)u]_x = 0, \\ S_t + u S_x - k^2 u_x = 0, \\ p = \Phi(\rho, I), \end{cases}$$

where ρ = density, u = velocity, p = pressure, S = component S^{11} of the stress deviation tensor, e = total specific energy, $I = e - u^2/2$ = internal specific energy, Φ is a function of two real variables and k^2 is a constant. In the case of a shock wave these variables are simultaneously discontinuous and so the term uS_x appears in the form of an ambiguous product of a discontinuous function by the derivative of another function discontinuous at the same point: in short, a product of the kind $Y \cdot \delta$, where Y = Heaviside function ($Y(x) = 0$ if $x < 0$, $Y(x) = 1$ if $x > 0$) and where δ is the Dirac mass at the origin ($\delta(x) = 0$ if $x \neq 0$, $\delta(0)$ so large such that $\int \delta(\lambda)\, d\lambda = 1$; one has intuitively $Y' = \delta$). The fourth equation in system (3) follows from Hooke's law stated as an infinitesimal linear stress—strain relationship in a frame of reference following the medium. For large deformations and strong collisions the usual linear form of Hooke's law **[5]** is no longer valid, and it is not known how to "escape" from the above problem of multiplication of distributions. Empirical numerical codes have been built for a few years by now for the numerical simulation of collisions: one observes shock wave solutions which reproduce the expected physical results. The same occurs in various other physical situations (shock waves in elastoplasticity, acoustics in a medium with piecewise C^∞ characteristics). Note that in the book under review the author does not consider explicitly (3), which can however be found in references, but the simplified model

$$(4) \qquad \begin{cases} u_t + u u_x = \sigma_x, \\ \sigma_t + u \sigma_x = k^2 u_x, \end{cases}$$

on which deeper mathematical results (solution of the Cauchy problem) can be obtained, but which is not exactly a system of physics. *As long as one considers that mathematics, physics and numerical tests have to fit together* (especially in the setting of partial differential equations) *this can be considered as a priori evidence for the need of an appropriate setting to define generalized solutions.*

In order to state the problem, the author begins with a review on some essentials in weak solutions and distributions. Among the difficulties which make the distribution setting inadequate, the most famous one is probably "Schwartz's impossibility result" (1954) [**14**]:

"Suppose given an associative algebra A with a derivative $D: A \to A$ (i.e., a linear map satisfying the rule $D(fg) = (Df) \cdot g + f \cdot (Dg)$). Suppose further that the space $\mathscr{C}(\mathbf{R})$ of continouous functions on $\mathbf{R}$ is included in A, that the derivative D induces on $\mathscr{C}^1(\mathbf{R})$ the usual derivative, that the function 1 is the unit element of A and finally that $\mathscr{C}(\mathbf{R})$ is a subalgebra of A. Then there is no element $\delta \in A$, $\delta \neq 0$, such that $x\delta = 0$."

If δ is the Dirac delta distribution (defined by the formula $\int \delta(x)\psi(x)\,dx = \psi(0)$ for any text function ψ) then one has $x\delta = 0$ in distribution theory (indeed $x\delta$ is defined in this theory by $\int x\delta(x)\psi(x)\,dx = (x\psi)(0) = 0$). Therefore the algebra A cannot contain the distribution δ and incorporate the above natural definition of the product $x\delta$. (Indeed and somewhat intuitively, the product $x\delta$ can be considered as null, but products such as $x\delta^2$, $x\delta^3, \ldots$ appear—when one approximates δ in a reasonable way—as being "infinite" while they should still be null in the algebra A as long as $x\delta = 0$.) The commonly accepted interpretation since 1954 is that "multiplication of distributions is impossible". Somewhat to anticipate—but also to clarify the ideas—the interpretation within the theory presented in Part II of the book under review, is that $x\delta$ is "infinitesimal" but not exactly null, permitting $x\delta^2, \ldots$ to be large, and the "impossibility" disappears. The philosophy of the "impossibility of the multiplication of distributions" has had far reaching consequences. Considerable effort has been invested in modifying the classical heuristic formulation of Quantum Field Theory, so as to avoid these products, but, even in this field, physicists go on introducing products of distributions. Also that philosophy certainly delayed the emergence of nonlinear theories capable of dealing with multiplications of distributions.

After a useful introduction (Part I) which—as far as the reviewer knows—has not been available in such a detailed and comprehensive form, the author undertakes the presentation of two nonlinear theories (Parts II and III respectively) introduced earlier in [**2, 3**] and [**12, 13**]. Form the author's introduction "The aim of this volume is to offer the reader a sufficiently detailed—yet easy—introduction to two of these recent nonlinear theories [...]. This introduction aims to bring the reader to the very level of ongoing research and equip him/her to pursue it if he/she wishes so. This may sound somewhat unlikely to those who are familiar with the rather lengthy and subtle technical intricacies of the linear theory of distributions [**15**, **7**]. However, as may well happen in the case of emergent theories, their strength can rather lie in the new ideas than in techniques, and of course, also in results these new ideas can bring about".

In the first of these two nonlinear theories the key to avoiding the "Schwartz impossibility result" lies in the consideration of two concepts of equality: a strong one (denoted by $=$) and coherent with all operations, including multiplication, and a weak one (called "association" and denoted by $\approx$) in general incoherent with the multiplication. Two "generalized functions" can be associated with each other, yet be different, but two associated distributions are equal. One has $x\delta \approx 0$ and $x\delta \neq 0$, and in this way the Schwartz impossibility result is circumvented. This is in complete agreement with classical analysis, when one has understood the essence of the matter: the new objects are more refined than the classical concept of a function and a distribution: the difference between $x\delta$ and 0 is "infinitesimal" (in the sense $x\delta \approx 0$) but not exactly null (since $x\delta \neq 0$). In classical analysis an infinitesimal quantity is null, so that this more refined difference does not make sense. Any theory dealing with products of singular objects (whose singularities lie in the same point) has to deal with "infinitesimal" and "infinitely large" quantities. In this sense the theory reminds one of a kind of Nonstandard Analysis. In the theory in Part II one has a canonical embedding of the distributions into the "generalized functions". This gives a canonical product of distributions. It is shown (book under review, Oberguggenberger [**10**, **11**]) that one obtains in this way a synthesis of most earlier known multiplications of distributions: Hörmander, Ambrose, Antosik-Mikusinski-Sikorski, Hirata-Ogata, Kaminski. A nice characterization in terms of Lojasiewicz sections is given in [**6**]. In the theory in Part II one "succeeds in proving quite impressive existence, uniqueness, regularity results concerning generalized solutions of linear and nonlinear partial differential equations". Concerning generalized solutions of partial differential equations, the genuine difficulty is often shifted to the final task of ascertaining that the generalized solution is indeed a classical function. This has been achieved within this theory, for systems of equations of physics and engineering such as (3) and (4) which do not have discontinuous solutions in the sense of distribution theory.

Part III introduces the earlier, more general, theory of the author [**12**, **13**]. Here one has only one kind of equality, but one cannot have a single differential algebra containing the distributions: one has to deal with a chain of such algebras, with partial differential operators mapping an algebra into another one. This reminds one of the classical chain of algebras $\mathscr{C}^m(\Omega)$ of functions of class C^m, $m = 1, 2, 3, \ldots$. This setting looks convenient, since a PDE usually involves only a finite number of partial derivatives. In his presentation the author has sought for greater generality: this theory "has so far concentrated on the most general algebraic and differential aspects of possible nonlinear theories of generalized functions, with the primary view of their use in the solution of rather arbitrary nonlinear partial differential equations, where in addition to the usual problems of existence, uniqueness and regularity of generalized solutions, the problems of stability, generality and exactness of such solutions have been emphasized". An interpretation of the classical Cauchy-Kovalevskaia theorem within this theory yields existence of generalized solutions which—with the possible exception of closed, nowhere dense subsets which may even have zero Lebesgue measure—are analytic on the whole of the domain of definition of the respective analytic equations. It is probably too early for a deep understanding of this kind of result, as well as many other ones presented in the book, such as existence results for linear PDEs with C^∞ coefficients, unsolvable within distribution theory (the Lewy equation for instance). Undoubtedly, these applications should be studied more in detail, especially in the light of particular cases of systems of equations stemming from physics and in the light of numerical experimentation.

Note that such a more detailed study has recently been done for discontinuous solutions of systems of equations in nonconservation form, and has been extremely fruitful. See below.

This book is not a classical textbook; it is resolutely a research book, in which the author expresses his own conclusion and his own conviction on a field which is emerging and is in rapid evolution. The problem itself is quite controversial. In the last 20 years, I have often heard prominent theoretical physicists telling that the mathematical difficulties connected with "multiplications of distributions in physics" were certainly one of "our epoch's main problems" in mathematics. On the other hand I have also often heard prominent mathematicians asserting that the problem had been completely settled in the negative by Schwartz's impossibility result. For those who believe in a somewhat "religious sense" that mathematics and physics are not dissociated, and that dissociations are only the superficial appearance of provisionally incoherent research developments, a satisfactory reconciliation is expected. *The book under review offers a significant contribution towards such a reconciliation, since it shows basic and general ways in which singular operations such as arbitrary products of distributions can be incorporated into mathematics, and then effectively applied to the solution of nonlinear PDEs.*

A criticism is that the book does not present the new numerical results that emerge from experiments and can be dealt with by this mathematical theory, although some references to engineering applications are given. Since the time this book was written (1985), a vast number of such applications of products of distributions to continuum mechanics have been obtained, for instance in elasticity, elastoplasticity and acoustics. These applications are presented in the more recent book [**1**] as well as in a series of preprints by the reviewer and collaborators.

As a matter of faith, or coming from their own experience, some mathematicians believe that all "correctly posed" problems of physics are in conservation form, and so, they never show products of distributions even in the case of shock waves. Therefore they do not accept motivations like those for instance connected with the system (3). However, even for such mathematicians, the mathematics in the book under review brings substantial numerical applications which take the form of new numerical schemes, useful for the classical system of fluid dynamics as well; see [**1**].

I recommend the reading of the book under review to those pure mathematicians who like to think calmly but deeply about the mathematical foundation of the problem. I believe that the presentation and style are excellent. However, the set of references is incomplete, especially for the theory in Part II, although it covers the material used by the author, and so is sufficient in this sense. An up to date and more complete set of references can be found in [**1**].

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J. F. Colombeau
Université de Bordeaux

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 1, January 1989

0273-0979/89 $1.00 + $.25 per page

Equivariant K-theory and freeness of group actions on C^*-algebras, by N. Christopher Phillips. Lecture Notes in Math., vol. 1274, Springer-Verlag, Berlin and New York, viii + 371 pp., $32.90. ISBN 3-540-18277-2

K-theory for C^*-algebras is also known under the name of "noncommutative" topology. A C^*-algebra is a Banach algebra that has the same abstract properties as the algebra $\mathscr{C}(X)$ of continuous complex-valued functions on a compact space X except for the fact that the multiplication is not necessarily commutative.

Noncommutative C^*-algebras arise naturally from group actions on topological spaces, foliated manifolds, pseudodifferential operators, etc., and they also formalize the noncommuting variables of quantum mechanics.

Even if one is only interested in spaces, one often has to extend the frame to the noncommutative category as certain natural constructions in K-theory automatically lead to noncommutative algebras. One might go as far as to compare this to the passage from real to complex numbers in analysis.