6. M. Gromov, *Partial differential relations*, Ergeb. Math. Grenzgeb., vol. 9, Springer-Verlag, Berlin and New York, 1986.

7. V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge Univ. Press, Cambridge, 1986.

WILLIAM M. BOOTHBY
WASHINGTON UNIVERSITY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 1, January 1989

0273-0979/89 $1.00 + $.25 per page

Commutator theory for congruence modular varieties, by Ralph Freese and Ralph McKenzie. London Mathematical Society Lecture Notes, vol. 125, Cambridge University Press, Cambridge, New York and Melbourne, 1987, 227 pp., $27.95. ISBN 0-521-34832-3

Ralph Freese and Ralph McKenzie are two of the outstanding North American contributors to the commutator; and after they have spent years revising and polishing preliminary manuscripts we finally have their fine book on the subject. The theme of the book is to show that the commutator is a versatile and useful tool; the authors succeed admirably in doing this. Roughly speaking, the commutator allows one to extend results for groups, rings and modules into the much larger domain of congruence modular varieties. Perhaps the best way to tell about the commutator is to sketch the background, and some of the results that have been achieved with it.

eUniversal algebra is mainly concerned with the study of algebraic structures (e.g., groups, lattices, rings, etc.), and with classes of such algebras defined by equations (e.g., groups of exponent 6, distributive lattices, idempotent rings, etc.). Some of the earliest results include the straightforward generalizations of the homomorphism theorems of group theory and ring theory. In the mid 1930s Birkhoff [2] made two fundamental contributions. First he showed that equational classes (classes of algebras defined by equations) were the same as varieties (classes of algebras closed under the formation of subalgebras, direct products and homomorphic images). Secondly he pointed out the importance of the lattice of congruences of an algebra. A congruence of an algebra **A** is an equivalence relation such that the obvious definition of a quotient algebra works; congruences can also be thought of as kernels of homomorphisms. The congruences of an algebra form a partially ordered set (under inclusion) which is a lattice.

In the 1960s and 1970s universal algebra received considerable stimulus from the tools and directions of logic. Jónsson [7] showed that one could use the ultraproduct construction from model theory to gain enormous insight into congruence distributive varieties, that is, varieties for which each of the algebras in the variety has a lattice of congruences satisfying the distributive law. The variety of lattices is probably the best known congruence distributive variety. Jónsson's work inspired many investigations—one of the most celebrated results is Baker's finite basis theorem **[1]** which says that every congruence distributive variety which is generated by finitely many finite algebras and has a finite language is finitely axiomatizable. The results of this period in the development of universal algebra did not usually connect with the traditional strongholds of algebra: groups, rings and modules. One might well ask if the word *universal* only applied to elementary results such as the isomorphism theorems. But this was soon to change.

In 1976 Smith published his startling Mal′cev Varieties **[10]** in which he showed that the notion of the center of a group could be generalized to varieties with permuting congruences (such varieties include groups, rings and modules)—he was particularly interested in applying his ideas to quasigroups. But the most amazing insight he had was that the commutator could be extended to such varieties. The commutator was not defined for pairs of elements, but for pairs of congruences θ, ϕ, yielding $[\theta, \phi]$. This was an extension of the commutator $[M, N]$ of two normal subgroups of a group. Suddenly the familiar words of group theory—center, Abelian, solvable, nilpotent, etc.—were available in a much wider context. Hagemann and Herrmann in Darmstadt were among the first to realize the possibilities of Smith's work. They promptly worked through his book and continued on to prove **[6]** that the commutator could be realized in arbitrary congruence modular varieties (i.e., equationally defined classes each of whose members has a congruence lattice which satisfies the modular law of Dedekind). An elegant geometric presentation of the commutator was then given by Gumm **[5]**; and also numerous details were reworked and improved by Taylor **[11]**.

In **[3]** Burris and McKenzie use the commutator to find a structure theorem for locally finite congruence modular varieties with a decidable first-order theory. The question of decidability is reduced to the case of modules for a finite ring (which is still open), and to some special varieties called discriminator varieties. This showed that universal algebra had developed the tools to make basic contributions to the study of decidability. (The restriction to congruence modular varieties has recently been eliminated by McKenzie and Valeriote **[9]**.) Over the years a number of researchers hoped to generalize Baker's Theorem mentioned above. The first real success was attained by McKenzie **[8]** using the commutator. Also let us mention the deep study of residually small congruence modular varieties made by Freese and McKenzie in **[4]**. Using the commutator they were able to estimate the sizes of subdirectly irreducible and simple algebras in such varieties.

This book is primarily aimed at researchers and students of universal algebra. The text has numerous exercises in it; however, since advanced courses in universal algebra are not widely available one cannot assume the readers have been exposed to the methods of proof needed. The authors have wisely included the solutions to the exercises at the end of the book. Also at the end of the book the reader will find an excellent survey of related literature, and a very useful index.

References

1. K. A. Baker, *Finite equational bases for finite algebras in a congruence distributive equational class*, Adv. in Math. **24** (1977), 207–243.

2. G. Birkhoff, *On the structure of abstract algebras*, Proc. Cambridge Philos. Soc. **31** (1935), 433–454.

3. S. Burris and R. McKenzie, *Decidability and Boolean representations*, Mem. Amer. Math. Soc., vol. 32, no. 246, 1981.

4. R. Freese and R. McKenzie, *Residually small varieties with modular congruence lattices*, Transl. Amer. Math. Soc. **264** (1981), 419–430.

5. H.-P. Gumm, *Geometrical methods in congruence modular algebras*, Mem. Amer. Math. Soc., vol. 45, no. 286, 1983.

6. J. Hagemann and C. Herrmann, *A concrete ideal multiplication for algebraic systems and its relation to congruence distributivity*, Arch. Math. (Basel) **32** (1979), 234–245.

7. B. Jónsson, *Algebras whose congruence lattices are distributive*, Math. Scand. **21** (1967), 110–121.

8. R. McKenzie, *Finite equational bases for congruence modular varieties*, Algebra Universalis **24** (1987), 224–251.

9. R. McKenzie and M. Valeriote, *The structure of decidable locally finite varieties*,

10. J. D. H. Smith, *Mal'cev varieties*, Lecture Notes in Math., vol. 554, Springer-Verlag, Berlin and New York, 1976.

11. W. Taylor, *Some applications of the term condition*, Algebra Universalis **14** (1982), 11–24.

Stanley Burris
University of Waterloo

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 1, January 1989

0273-0979/89 $1.00 + $.25 per page

Generalized solutions of nonlinear partial differential equations, by Elemér E. Rosinger. Mathematical Studies, vol. 146, North-Holland, Elsevier Science Publishers, Amsterdam, 1987, xvii + 409 pp., US $ 92.00; Dfl 175.00. ISBN 0-444-70310-1

Why the need for generalized solutions of partial differential equations? It has been recognized that many equations of physics do not have classical solutions (for instance shock wave solutions of systems of conservation laws). Distribution solutions—usually called "weak solutions"—of the model equation

$$u_t + uu_x = 0$$

are defined as those integrable functions u which satisfy: $\forall\psi \in \mathscr{C}^\infty(\mathbf{R}^2)$ with compact support

$$(1) \qquad \iint \left[u(x,t)\frac{\partial}{\partial t}\psi(x,t) + \frac{1}{2}u^2(x,t)\frac{\partial}{\partial x}\psi(x,t)\right] dx\,dt = 0.$$

In the case of linear equations a detailed theory has been developed [**15**, 7]. However the situation is far from being satisfactory. Lewy [**8**] showed that the very simple linear equation

$$(2) \qquad \frac{\partial}{\partial x_1}u + i\frac{\partial}{\partial x_2}u - 2i(x_1 + ix_2)\frac{\partial}{\partial x_3}u = f$$