BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 1, January 1989

0273-0979/89 $1.00 + $.25 per page

Symplectic geometry and analytical mechanics, by Paulette Libermann and Charles-Michel Marle. Translated by Bertram Eugene Schwarzbach. Mathematics and Its Applications, D. Reidel Publishing Company, Dordrecht, Boston, Lancaster, Tokyo, 1987, xvi + 526 pp., $89.00. ISBN 90-277-2438-5

This is the most recent of several books appearing over the last fifteen years or so which well may help to restore analytical mechanics to the important place it once occupied in the training of mathematicians. Courses in this subject, then known as "rational mechanics", were standard fare in many mathematics departments until about 50 years ago, at which time they virtually disappeared and the subject went underground for a generation of mathematicians—at least as far as graduate study in the United States was concerned. Of course, it did not disappear as a topic of research, and in fact this spate of excellent books has surely resulted in large part from the profound and exciting research which has been done in analytical mechanics or has been suggested by it over this same period, and in some part too from the tremendous advances in related manifold theory and differential geometry, which have revolutionized the approach to mechanics and made much of this work possible. Happily there now exist several extremely well written and carefully designed texts from which a graduate student or working mathematician can learn this important subject, the well-spring of so much mathematics, and can even come up to the very frontiers of research. Of these the books of Arnold [**1**] and Abraham and Marsden [**2**] may be the most frequently used and best known (in English). To this I would add one of the earliest of this breed, Godbillon [**3**], which like the other two is extremely clear and well written but which because of its more limited scope has the advantage (especially for the beginner who wants a quick start) of being very much shorter. The present book of Libermann and Marle, which is based on courses given to graduate students at the University of Paris VI and VII, is very similar in level and "prerequisites" to these three. In style and treatment it is most like Godbillon, but in scope and content much closer to Arnold and Abraham and Marsden, with both of which it has a great deal of overlap. But there are also important differences. Its emphasis is very much on the mathematical underpinnings of mechanics and, with one exception, there is almost no discussion of physical examples as there is in both Abraham and Marsden and, especially, Arnold. Thus it has its own style, approach, and priorities, which give it an emphasis different from the others. To me this makes it a useful complement to other books with the same objective—making analytical mechanics and its related research available to the mathematically trained—and I find it a welcome addition to the collection of books with this goal.

In its modern reincarnation, more than ever before, mastery of this subject requires an interest in or willingness to learn modern differential geometry and manifold theory. To one who already has some knowledge of these subjects, analytical mechanics serves as a fascinating and concrete example of many concepts which may have been learned in a rather abstract setting, and as an important lesson in the historical background of manifold theory (which owes so much to Poincaré's work in mechanics). To one who is not already skilled in modern differential geometry, these books present well-arranged and carefully thought out introductions to the subject in a highly motivated context.

Really excellent reviews of [**1** and **2**] by reviewers who are themselves important contributors to the subject have appeared in this journal [**4**]. They contain a great deal of historical background and, in particular, Sternberg's review of [2] contains an interesting account of the main themes of mathematical research, which I mentioned above as having so revitalized this classical subject. These reviews should certainly be read by anyone who wants to get a feel for the available texts up to 1980 before launching into a study of analytical mechanics. It would be superfluous and even presumptuous of me to attempt to cover the same material, so I will limit myself to just enough background to explain some of the emphasis of the book being reviewed and to try to contrast its contents with the others.

Let us consider for a moment a system of m particles in Euclidean space. Their positions are given by a point in $M = R^n$, $n = 3m$, a differentiable manifold, the *configuration* space. The collection of all tangent vectors at all points of R^n is just $R^{2n} = R^n \times R^n$ with n coordinates for the initial point of the vector and n for its components. This is again a manifold $T(R^n)$, the *tangent bundle* of R^n. A vector field X on R^n is a mapping $X\colon R^n \to T(R^n)$ assigning to each X a vector $X(x) \in T_x(R^n)$, the vector space of all vectors with x as initial point. The vector field X is the geometric counterpart of a system of first order ordinary differential equations, $\dot{x}_i = f_i(x)$, $i = 1, \dots, n$, on R^n, the right sides being the components of the vector field relative to the usual orthonormal basis of R^n. The solutions of the equations correspond to the integral curves of the vector field. Now the laws of motion, say, of a system of particles in a gravitational field, would be given by a system of *second* order differential equations, $\ddot{x}_i = h_i(x)$, $i = 1, \ldots, n$, but this can be replaced by a first order system by introducing the first derivatives as additional variables, thus obtaining the equivalent system $\dot{x}_i = v_i$; $\dot{v}_i = h_i(x)$, $i = 1, \ldots, n$. This in turn can be considered as a vector field Z on $T(R^n)$. According to the existence theorem for differential equations, the system's configuration and its velocity, $x(t; x_0, v_0)$ and $v(t; x_0, v_0) = \dot{x}(t; x_0, v_0)$, i.e. the *state* of the system at any time t, are completely determined by the position and velocity (x_0, v_0) at time $t = 0$ and the differential equations, or the vector field Z, on the manifold $T(R^n)$, the *state* or *phase space* of the system of m points.

If all mechanical systems corresponded to R^n, as they do in fact locally, then geometry would play a less important role. But other manifolds M are required to describe the configurations of even simple mechanical systems: the sphere $S^2 = M$ for a "spherical pendulum", whose bob is free to move in any direction, the torus $T^2 = M$ for a "double pendulum"—a plane pendulum whose arm is hinged at the top and at some other point as well, $M = \mathrm{SO}(3)$ for a rigid body with one point fixed, etc. In each case the phase space is the tangent bundle $T(M)$ and the motion is completely determined by the integral curves of a vector field Z on $T(M)$. This description can be termed the Lagrangian point-of-view. We already see here the need for manifold theory. But a second reason why modern differential geometric techniques are so important is that a good presentation of Hamilton's approach to mechanics requires that we consider a different phase space although the configuration space remains the same. Just as any vector space V of dimension n over R has a dual space V^* whose elements are linear mappings of V into R, so too, we may attach to each point of R^n, or indeed of any manifold M, a dual space $T_x^*(M)$ to the space $T_x(M)$ of tangent vectors at $x \in M$. The collection of all of these dual spaces is again a manifold $T^*(M)$, the *cotangent* bundle, an object that, like the dual space of a vector space, is somehow much harder to visualize. Once again the laws of motion for the particular system are embodied in a vector field Z on the phase space $T^*(M)$ with its evolution in time corresponding to the integral curve through the point representing its initial state. Surprisingly however the geometry of these two phase spaces is very different—there is a naturally or canonically defined skew-symmetric 2-form Ω_x, of maximum rank $(= 2n)$ defined on the (dual) vector space $T_x^*(M)$ at each $x \in M$, i.e. a natural exterior 2-form Ω on $T^*(M)$, which in this case is itself the exterior derivative $d\omega$ of a natural 1-form ω on $T^*(M)$. This Ω is called a *symplectic* form and defines a "symplectic geometry" on the manifold $T^*(M)$ with many interesting properties. Ω is defined quite independently of the mechanical system: it exists automatically on the cotangent bundle of any manifold whatsoever whether or not it has arisen as the phase manifold of some mechanical system. But in the present instance the geometry is interwoven with that of the mechanical system: it is invariant under the flow defined by a vector field Z, known as a Hamiltonian vector field, which embodies the dynamics of the system. The relationship of the system and the symplectic geometry make it possible to gain important new insights about the system and to solve problems which could not be solved in the pre-Hamilton approach. It is astounding that this should be so—a vector space and its dual, having the same dimension, are isomorphic, and one would expect $T(M)$ and $T^*(M)$ to have isomorphic geometries, but in fact they do not and the natural arena for the study of $T^*(M)$ and its symplectic form Ω, as well as the related *contact* form ω, is the geometry of differential forms on a manifold, essentially the creation of É. Cartan. Of course, none of this differential geometry was available at the time of Hamilton, but now it is and it serves as an essential tool in interpreting classical theorems, solving hitherto unsolved problems and posing new and interesting ones. It also reveals important connections with other areas of mathematics, group representations for example, as in the work of Kirillov and Kostant (see **[4]** for a discussion and references). In fact the study of manifolds of even dimension equipped with a closed exterior 2-form Ω of maximum rank, i.e. *symplectic manifolds*, has become an important area of research in differential geometry, global analysis, and in theoretical physics, where this seems to be a very natural structure (see, for example, important work of Gromov as discussed in the Séminaire Bourbaki article of Bennequin **[5]** or in the recent book of Gromov himself **[6]** and the second edition of Guillemin and Sternberg [7]—both of the latter reviewed recently in this Bulletin). The application of manifold theory and differential geometry to mechanics, especially to qualitative questions, was originated by Poincaré, but the study of Hamiltonian mechanics owes much to the above-mentioned work of Cartan.

Thus we see that any book or course which has the purpose of making analytical mechanics accessible to the general mathematical reader must first of all present the differential geometric foundations—vector fields on manifolds, vector bundles over manifolds, the geometry of exterior differential forms, symplectic geometry, etc.—as carefully and understandably as possible. This is done across an interesting spectrum of approaches in each of the books mentioned. At one end is Arnold, the most concrete and example-oriented: the mathematical concepts are introduced only as they are needed to develop the examples from mechanics. At the other end Libermann and Marle have very few mechanical examples, and then only after the mathematics has been very thoroughly developed. Abraham and Marsden is in the intermediate position. Which approach one prefers is very much a matter of personal taste and objectives, the way one prefers to learn a new subject, and where one is starting from. I find it very pleasant to have the choice of (at least !) four well-written books to skip around in.

With this introduction we turn to a brief description of Libermann-Marle's book itself. It contains five chapters (335 pp.) to which have been added seven appendices (160 pp.) and a thirty page bibliography. Of the three chapters written by Libermann (I, II, V) the first contains a very complete presentation of symplectic geometry, vector bundles, and exterior differential forms. As might be expected of a distinguished practitioner of exterior calculus, there is much more on this subject than can be found in any of the other books on mechanics, e.g. the LePage decomposition theorems in the first chapter and then a complete chapter, Chapter V, on contact forms and structures. Chapter II puts all of this in the setting needed for mechanics, discussing variational principles, deriving Lagrange's and Hamilton's equations and explaining the important concept of a Legendre transformation, which forms the bridge between the Lagrange and the Hamilton approach, i.e. between $T(M)$ and $T^*(M)$. None of this basic material is easy, even for someone with a general knowledge of manifold theory, but the exposition of the theory here is clear and careful. There is, to be sure, extensive overlap with the presentation in other texts such as those mentioned above, but there is also a good deal of interesting material not found in them. In Chapters III and IV, written by Marle, we find a detailed study of symplectic manifolds, Poisson manifolds, the Darboux-Weinstein theorems and a wealth of related material. Chapter IV is completely devoted to the action of a Lie group on a symplectic manifold. It covers, in particular, the momentum map of Souriau and Smale and leads into the work of Kirillov, Kostant and Souriau mentioned earlier and ends with an analysis of the motions of a rigid body in several specific cases—the Euler-Poinsot, Euler-Lagrange and Kowalevska problems. Both of these chapters have problem sets.

The seven appendices contain some background material on basic differential geometry, distributions and pfaffian systems, Frobenius Theorem (as generalized by Stefan and Sussmann), foliations, and Lie groups, for example. But they also contain several important supplements to the text: jets, Lagrange-Grassmann manifold, Morse families and Lagrangian submanifolds, and an appendix on integral invariants. There are a number of interesting comments and historical notes throughout the text, including some discussion of topics which have been omitted accompanied by references to the relevant literature. A study of this book, or even selected parts of it, will surely bring the reader to the point at which the research articles listed in the extensive bibliography are quite accessible, which is indeed one of its aims. Finally, it should be mentioned that although originally written in French, the translation by B. E. Schwarzbach is very good and the exposition throughout is thorough and careful. This book is a welcome addition to the literature. It will surely prove useful as a reference, a place to learn the subject, or as an adjunct to the earlier books mentioned above.

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AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 1, January 1989

0273-0979/89 $1.00 + $.25 per page

Commutator theory for congruence modular varieties, by Ralph Freese and Ralph McKenzie. London Mathematical Society Lecture Notes, vol. 125, Cambridge University Press, Cambridge, New York and Melbourne, 1987, 227 pp., $27.95. ISBN 0-521-34832-3

Ralph Freese and Ralph McKenzie are two of the outstanding North American contributors to the commutator; and after they have spent years revising and polishing preliminary manuscripts we finally have their fine book on the subject. The theme of the book is to show that the commutator is a versatile and useful tool; the authors succeed admirably in doing this. Roughly speaking, the commutator allows one to extend results for groups, rings and modules into the much larger domain of congruence modular varieties. Perhaps the best way to tell about the commutator is to sketch the background, and some of the results that have been achieved with it.

eUniversal algebra is mainly concerned with the study of algebraic structures (e.g., groups, lattices, rings, etc.), and with classes of such algebras defined by equations (e.g., groups of exponent 6, distributive lattices, idempotent rings, etc.). Some of the earliest results include the straightforward generalizations of the homomorphism theorems of group theory and ring theory. In the mid 1930s Birkhoff [2] made two fundamental contributions. First he showed that equational classes (classes of algebras defined by equations) were the same as varieties (classes of algebras closed under the formation of subalgebras, direct products and homomorphic images). Secondly he pointed out the importance of the lattice of congruences of an algebra. A congruence of an algebra **A** is an equivalence relation such that the obvious definition of a quotient algebra works; congruences can also be thought of as kernels of homomorphisms. The congruences of an algebra form a partially ordered set (under inclusion) which is a lattice.

In the 1960s and 1970s universal algebra received considerable stimulus from the tools and directions of logic. Jónsson [7] showed that one could use the ultraproduct construction from model theory to gain enormous insight into congruence distributive varieties, that is, varieties for which each of the algebras in the variety has a lattice of congruences satisfying the distributive law. The variety of lattices is probably the best known congruence