BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 20, Number 1, January 1989

COMPACT MANIFOLDS WITH A LITTLE NEGATIVE CURVATURE

K. D. ELWORTHY AND S. ROSENBERG

1. Bochner's Theorem states that a compact oriented Riemannian manifold (M, g) with positive Ricci curvature has $H^1(M;\mathbf{R}) = 0$. Myers' Theorem implies the stronger result that $\pi_1(M)$ is finite under the same hypothesis. Both theorems fail if the Ricci curvature is positive except on a set of arbitrarily small diameter, since every compact manifold admits such a metric of volume one. Nevertheless, we can extend these theorems and the Bochner Theorem for p-forms, yielding topological obstructions to manifolds admitting metrics with a little negative curvature.

2. Results for $H^1(M;\mathbf{R})$. The Laplacian on p-forms has the Weitzenböck decomposition $\Delta^p = \nabla^*\nabla + R^p$; here ∇ is the Levi-Civita connection and $R^p \in \mathrm{End}(\Lambda^p T^*M)$ with $R^1 = \mathrm{Ricci}$. We write $R^p(x) \geq R_0$ for $x \in M$ if the lowest eigenvalue of $R^p(x)$ is at least R_0. In what follows, we normalize all metrics to have volume one.

THEOREM 1. *Pick $R_0 > 0$ and $K < 0$. There exists $\varepsilon = \varepsilon(R_0, K, \dim M) > 0$ such that if $\mathrm{Ric}(x) \geq R_0$ except on a set A, with diameter $\mathrm{diam}(A) \leq \varepsilon$, where $\mathrm{Ric}(x) \geq K$, then $H^1(M;\mathbf{R}) = 0$.*

In other words, if the metric has a deep well of negative Ricci curvature, we may still conclude $H^1(M;\mathbf{R}) = 0$ provided the well is narrow enough. Notice that there is no restriction on the topology of A.

Theorem 1 is a consequence of the following weaker version about metrics with a shallow well of negative Ricci curvature.

THEOREM 1′. *Pick $R_0 > 0$. There exists $\varepsilon' = \varepsilon'(R_0, \dim M) > 0$ and $\delta = \delta(R_0, \dim M) < 0$ such that if $\mathrm{Ric}(x) \geq R_0$ except on a set A, with $\mathrm{diam}(A) \leq \varepsilon'$, where $\mathrm{Ric}(x) \geq \delta$, then $H^1(M;\mathbf{R}) = 0$.*

We sketch a proof of Theorem 1′. By semigroup domination for the heat flow on one forms, it is enough to show that $\Delta^0 + \mathrm{Ric}' > 0$, where $\mathrm{Ric}'(x)$ is the lowest eigenvalue of Ricci at x. By an elementary argument, we have

LEMMA 2. *Let $V\colon M \to \mathbf{R}$ be continuous. If* (i) $\int_M V\, d\mathrm{vol}(g) > 0$ *and* (ii) $\lambda_1 \geq -V_{\min} + \frac{\|V - V_{av}\|^2}{\int_M V}$, *then $\Delta^0 + V > 0$.*

Received by the editors February 15, 1988 and, in revised form, July 7, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53C20; Secondary 58G11, 58G32, 58C40.
The first author was partially supported by the SERC and the second author by the NSF.

Here λ_1 is the first nonzero eigenvalue of Δ^0, $V_{\min}$ is the minimum of V, $V_{av} = \text{vol}(M)^{-1} \int V$ and $\|\cdot\|$ is the L^2-norm. We set $V = \min\{R_0, \text{Ric}'\}$. Then for ε' and δ sufficiently small, (i) holds and the right side of (ii) is arbitrarily close to zero. However, by Myers' Theorem, the diameter of $M - A$ and hence of M is bounded above. By Gromov **[5]** or Li and Yau **[8]**, this keeps λ_1 bounded away from zero as ε', δ go to zero. Thus $\Delta^0 + V > 0$ and hence $\Delta^0 + \text{Ric}' > 0$.

To derive Theorem 1, we strengthen Lemma 2. If $\Delta^0 f = \lambda_1 f$, then we apparently need $\lambda_1 \geq -V_{\min}$ to show $\langle(\Delta^0 + V)f, f\rangle > 0$. However, we can do much better provided f is not concentrated near $V_{\min}$. In fact, by estimates of Li **[7]** and Croke **[2]** we can estimate how concentrated any function in the span of the first m eigenfunctions of Δ^0 may be near $V_{\min}$. Roughly speaking, this allows $V_{\min} = \text{Ric}'_{\min}$ to be arbitrarily negative and to replace λ_1 by λ_m in Lemma 2(ii). Now we can mimic the proof of Theorem 1′ using the estimates in Li and Yau **[9]** for λ_m. The method of proof yields explicit upper bounds for $\varepsilon, \varepsilon'$, and $|\delta|$ in terms of the geometric data.

A different method of coupling geometric information with semigroup domination may be found in **[1]**.

3. Results for $\pi_1(M)$. Here the results for deep and shallow wells differ.

THEOREM 3. *Assume M admits a metric g with* $\text{Ric}(x) \geq R_0 > 0$ *except on a set A, with* $\text{diam}(A) \leq \varepsilon$, *where* $\text{Ric}(x) \geq K$, *for ε as in Theorem* 1. *If $\pi_1(M)$ contains a solvable subgroup of finite index, then $\pi_1(M)$ is finite. In particular, if $\pi_1(M)$ has polynomial growth, then $\pi_1(M)$ is finite* **[4]**.

As opposed to Myers' theorem, the proof uses $H^1 = 0$ to show π_1 is finite. In the tower of coverings $\tilde{M} \to M_k \to M_{k-1} \to \cdots \to M_0 \to M$ associated to the solvable subgroup, we argue inductively that $H^1(M_j; \mathbf{R}) = 0$ implies M_{j+1} is a finite cover of M_j, noting that $\Delta^0 + \text{Ric}'$ is still positive for finite covers of M.

If a manifold with infinite π_1 admits a shallow well metric, the metric must be very distorted, in the sense that either the injectivity radius is very small at each point, or a generator of π_1 has very long geodesic length. To be more precise, we fix a point x_0 of M.

THEOREM 3′. *Suppose $\pi_1(M, x_0)$ is infinite. For a set of generators $G = \{\gamma_1, \ldots, \gamma_t\}$ for $\pi_1(M, x_0)$ and for positive numbers l, ρ and R_0, there exist $\delta = \delta(R_0, \dim M, G, l, \rho, \pi_1(M, x_0)) < 0$ and $\varepsilon = \varepsilon(R_0, \dim M) > 0$ such that if g is a metric satisfying*

(i) *some point of M has injectivity radius larger than ρ,*

(ii) *the shortest geodesic in γ_i has length less than l for each i,*

(iii) $\text{Ric}(g) \geq R_0$ *except on a set of diameter less than ε,*

then $\text{Ric}(g) < \delta$ *somewhere on M.*

Here we bound the growth function $\gamma(r)$ of π_1 by $C_1 \cdot \exp(C_2\sqrt{-\delta}r)$ for positive constants C_1, C_2 as in **[4, 11]**. For fixed $C_3 > 0$ and $N \in \mathbf{Z}^+$, this is bounded in turn by $C_3 \cdot r$ for $r = 1, 2, \ldots, N$ by taking δ close to zero.

For N sufficiently large, this implies $\pi_1(M)$ contains a nilpotent subgroup of finite index [4] and Theorem 3 applies.

4. Results for p-forms. $H^p(M, \mathbf{R}) = 0$ if R^p is positive. More generally, if we define $R^{p'}$ analogously to Ric$'$, then $H^p(M, \mathbf{R}) = 0$ whenever $\nu^p = \overline{\lim}_{t\to\infty} t^{-1} \ln \mathbf{E}[\exp(-\int_0^t R^{p'}(x_s)\, ds)] < 0$. Here $\mathbf{E}$ is expectation with respect to the Wiener measure for Brownian motion x_s on M. For the universal cover $\tilde{M}$, $\nu^p(M) = \nu^p(\tilde{M})$ with the pullback metric, so $\nu^p < 0$ implies the vanishing of the space of L^2 harmonic p-forms on $\tilde{M}$. By the weak Hodge Theorem, $\mathrm{Im}[H_c^p(\tilde{M}; \mathbf{R}) \to H^p(\tilde{M}; \mathbf{R})] = 0$, where H_c^p denotes cohomology with compact supports. This implies that no nonzero class in $H^p(\tilde{M}; \mathbf{R})$ has a representative differential form with compact support. For $p = 1$, we showed in [3] that in fact $H_c^1(\tilde{M}; \mathbf{Z}) = 0$, so in particular a compact 3-manifold with infinite π_1 and admitting a metric as in Theorem 3 must be a $K(\pi, 1)$.

For higher dimensional manifolds, we fix generators of $\pi_1(M)$ with associated growth function $\gamma(r)$ and a function $f(r)$ with $\limsup_{r\to\infty} f(r)\gamma(kr) = 0$ for all $k \in \mathbf{Z}^+ \cdot f(r)$ is then independent of the choice of generators.

THEOREM 4. *Assume $R^p > 0$ or more generally that $\nu^p < 0$ on M. Let r denote the distance from a fixed point in $\tilde{M}$. If $\pi_1(M)$ is infinite, no nonzero class in $H^p(\tilde{M}; \mathbf{R})$ has a representative form which decays faster than $f(r)$.*

By Micallef-Moore [10], a simply connected manifold with curvature operator positive on complex totally isotropic two-planes is homeomorphic to a sphere ($\dim M \geq 4$). It is known that this curvature condition implies $R^2 > 0$ if $\dim M$ is even, and it may be that it implies $R^p > 0$ for $p \neq 1, n-1$. Thus Theorem 4 gives topological information on nonsimply connected manifolds with this type of curvature operator, at least for $p = 2$ and $\dim M$ even.

To prove Theorem 4, we use a notion of bounded homology H_p^∞ and l_1-cohomology H_1^p complementary to Gromov's bounded cohomology [6]. As in [3, Theorem 5A], the integral of a compactly supported closed p-form over a bounded chain is unchanged under the heat flow and decays to zero as $t \to \infty$, so $\mathrm{Im}[H_c^p \to H_1^p] = 0$. Using a Poincaré duality map in this theory and the fact that $\nu^p = \nu^{n-p}$, we conclude that every class $\alpha \in H_p(\tilde{M})$ is the boundary of an infinite chain $\sigma = \sum n_i \sigma_i$ with bounded coefficients. Let θ be a closed differential form which decays faster than $f(r)$. By estimates in [11], the boundary of suitable partial sums of σ has volume growth bounded by $\gamma(kr)$ for some k, so the integral of θ over the boundary of these partial sums tends to zero at infinity. Thus $\int_\alpha \theta = 0$ so θ is cohomologous to zero.

REFERENCES

1. P. Bérard and G. Besson, *Number of bound states and estimates for some geometric invariants*, Preprint.

2. C. Croke, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. École Norm. Sup. Paris **13** (1980), 419–435.

3. K. D. Elworthy and S. Rosenberg, *Generalized Bochner Theorems and the spectrum of complete manifolds*, Acta Math. Appl. (to appear).

4. M. Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. **53** (1981), 53–78.

5. ——, *Paul Levy's isoperimetric constant*, Inst. Hautes Études Sci. Publ. Math., preprint, 1980.

6. ——, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. **56** (1982), 5–100.

7. P. Li, *On the Sobolev constant and the p-spectrum of a compact Riemannian manifold*, Ann. Sci. École Norm. Sup. Paris **13** (1980), 451–469.

8. P. Li and S. T. Yau, *Estimates of eigenvalues on a compact Riemannian manifold*, Proc. Sympos. Pure Math., vol. 36, Amer. Math. Soc., Providence, R.I., 1980, pp. 205–240.

9. ——, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), 153–201.

10. M. Micallef and J. D. Moore, *Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes*, Ann. of Math. (2) **127** (1988), 199–227.

11. J. Milnor, *A note on curvature and fundamental group*, J. Differential Geom. **2** (1968), 1–7.

Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom

Department of Mathematics, Boston University, Boston, Massachusetts 02215