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## THE NONDEGENERATION OF THE HODGE-DE RHAM SPECTRAL SEQUENCE

HARSH V. PITTIE

**ABSTRACT.** Using an integrable, homogeneous complex structure on the compact group  $SO(9)$ , we show that the Hodge-de Rham spectral sequence for this non-Kähler compact complex manifold does *not* degenerate at  $E_2$ , contrary to a well-known conjecture.

On any (compact) complex manifold  $M$ , the algebra of global complex-valued  $C^\infty$ -differential forms  $a^*(M)$  has a bigrading given by the Hodge type; and the corresponding decomposition of the de Rham differential  $d = \partial + \bar{\partial}$  gives rise to a double complex  $(a^{*,*}(M), \partial, \bar{\partial})$ . The spectral sequence corresponding to the first ("holomorphic") degree is  $E_1^{p,q} = H^q(M, \Omega_M^p) \Rightarrow H_{DR}^{p+q}(M)$  with  $d_1 = \partial$ : this is the Hodge-de Rham (HdR) spectral sequence. When  $M$  is compact and Kähler,  $E_1 = E_\infty$  by Hodge theory. A folklore conjecture of about thirty years' standing says that for *any* compact complex manifold one should have  $E_2 = E_\infty$ . We will give an example to show that this conjecture is false.

1. There is an old observation of H. Samelson<sup>1</sup> (see Wang [4]) that every compact Lie group  $G$  of even dimension (equivalently even rank) can be made into a complex manifold in such a way that all left-translations by elements  $g \in G$  are holomorphic maps: we call such a structure an LICS on  $G$  (= left invariant, integrable complex structure on  $G$ ). If  $G$  is in addition semisimple, then no LICS can be Kählerian because  $H^2(G; \mathbf{R}) = 0$ . Our example is a particular LICS on  $SO(9)$  (equivalently  $Spin(9)$ : see §3 below), for which  $E_2$  has complex dimension 26. Since one knows that  $E_\infty$  has complex dimension 16, we obtain the required nondegeneration of HdR.

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<sup>1</sup>Samelson's paper appeared in a Portuguese journal, which is perhaps understandably not available in Indian libraries.

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This is the ‘first possible’ case among LICS on  $G$  for which  $E_2 \neq E_\infty$ , in the sense that it is easy to check that for the semisimple groups of rank 2, the conjecture is true (see §4 below).

2. The details of the proof sketched below are contained in a forthcoming manuscript [3]. To describe the main features we need to recapitulate some general facts about the moduli space of all LICS on a given  $G$ , which is itself a new result. (Compare Theorem C of Wang [4].)

Let  $\mathfrak{g}$  be the real Lie algebra of  $G$ , and  $\mathfrak{g}^c$  its complexification. Fix a Borel subalgebra  $\mathfrak{b} \subseteq \mathfrak{g}^c$ : the intersection  $\mathfrak{b} \cap \mathfrak{g}$  is a real maximal toral subalgebra  $\mathfrak{t}$  of  $\mathfrak{g}$ . Then  $\mathfrak{b}$  is described as an extension

$$(2.1) \quad 0 \rightarrow \mathfrak{n} \rightarrow \mathfrak{b} \xrightarrow{\pi} \mathfrak{h} \rightarrow 0$$

where  $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$  and the isomorphism  $\mathfrak{t} \otimes \mathbb{C} \hookrightarrow \mathfrak{b} \xrightarrow{\pi} \mathfrak{h}$  splits the extension. Let  $\mathfrak{a} \subseteq \mathfrak{h}$  be a complex subspace of half the dimension (recall the rank is even) such that  $\mathfrak{a} \cap \mathfrak{t} = \{0\}$ : equivalently, that  $\mathfrak{h} = \mathfrak{a} \oplus \bar{\mathfrak{a}}$ ,  $\bar{\mathfrak{a}}$  being the conjugate subspace of  $\mathfrak{a}$  with respect to  $\mathfrak{t}$ . The extension (2.1) restricted to  $\mathfrak{a} \subseteq \mathfrak{h}$  gives a solvable subalgebra  $\mathfrak{s} \subseteq \mathfrak{g}^c$  defined by

$$(2.2) \quad 0 \rightarrow \mathfrak{n} \rightarrow \mathfrak{s} \rightarrow \mathfrak{a} \rightarrow 0$$

which is again split; and the space of all LICS on  $G$  is essentially (up to a discrete group of outer automorphisms) the space of all solvable subalgebras of  $\mathfrak{g}^c$  of type (2, 2). Indeed, for the LICS corresponding to  $\mathfrak{s}$ ,  $G$  can be shown to be the homogeneous space of its complexification  $G^c$ , given by  $G = G^c/S$ , where  $S$  is the connected solvable subgroup of  $G^c$  corresponding to  $\mathfrak{s} \subseteq \mathfrak{g}^c$ .

3. It is easy to verify directly that  $H_{\mathfrak{g}}^z(G)$  is independent of *finite* coverings: hence for semisimple  $G$  we can use the Lie algebraic formula of Bott [1] (established for simply connected groups) that

$$(3.1) \quad H_{\mathfrak{g}}^{p,q}(G) = H^q(\mathfrak{s}, \Lambda^p(\mathfrak{m}))$$

where  $\mathfrak{m}$  is the  $\text{ad}(\mathfrak{s})$ -module  $(\mathfrak{g}^c/\mathfrak{s})^*$  (Theorem II of [1]).

Applying (2.2) and Theorem 13 of Hochschild and Serre [2] to the right-hand side of (3.1), we obtain

$$(3.2) \quad H^{p,q}(G) = \sum_{r \geq 0} H^r(\mathfrak{a}) \otimes H^{q-r}(\mathfrak{n}, \Lambda^p(\mathfrak{m}))^{\mathfrak{a}},$$

where the superscript  $\mathfrak{a}$  refers to the invariants under the action of  $\mathfrak{a}$  obtained from the splitting of (2.2). Since for any abelian Lie algebra we have  $H^*(\mathfrak{a}) = \Lambda^*(\mathfrak{a}^*)$ , our task is reduced to computing the second factor  $H^*(\mathfrak{n}, \Lambda^*(\mathfrak{m}))^{\mathfrak{a}}$ , which we call the *invariant cohomology* of  $G$  in the given LICS.

4. Our next main result says that this invariant cohomology is computed by the following bigraded differential algebra. Put

$$(4.1) \quad B^{i,j} = \Lambda^{j-i}(\bar{\mathfrak{a}}^*) \otimes H^{i,i}(G/T : \mathbb{C})$$

where  $T \subseteq G$  is a maximal torus. The differential  $\delta$  of type  $(1, 0)$  on  $B^{*,*}$  is specified by the requirements

- (a)  $\delta: \Lambda^1(\bar{\alpha}^*) \rightarrow H^{1,1}(G/T : \mathbf{C}) \simeq \mathfrak{h}^*$  is the inclusion,
- (b)  $\delta$  is a derivation of type  $(1, 0)$ .

Thus

$$(4.2) \quad H^{i,j}(B, \delta) = H^i(\mathfrak{n}, \Lambda^j(\mathfrak{m}))^{\mathfrak{a}}.$$

The “shear” in the bigrading given by (4.1) and (4.2) is essentially dictated by (3.2) and Serre duality.

This complex gives an immediate computation of  $H_{\bar{\theta}}(G)$  for all LICS on semisimple groups of rank 2; and it shows that Borel’s trigraded spectral sequence for the *holomorphic* fibre bundle  $T \rightarrow G \rightarrow G/T$  degenerates at  $E_3$  for any  $G$ . This last fact is the main reason for starting the computations with Bott’s formula (3.1).

5. With all this in hand, we now describe a particular LICS on  $\mathrm{SO}(9)$  whose Dolbeault cohomology ring is suitably “large”. Then computation in  $E_1$  of HdR gives an  $E_2$ -term whose total complex dimension =  $\sum_{p,q} \dim_{\mathbf{C}} E_2^{p,q} = 26$ , giving the counterexample as explained above. Further computation with *this example* shows that  $E_3 = E_{\infty}$  for HdR.

Let  $\mathfrak{t} \subseteq \mathfrak{so}(9)$  be a real maximal toral subalgebra and  $e_1, \dots, e_4$  an orthonormal basis with respect to a Weyl invariant positive inner product on  $\mathfrak{t}$ : and let  $x_1, \dots, x_4 \in \mathfrak{t}^*$  be the dual coordinate basis. The polynomials

$$f_k = \sum_{j=1}^4 x_j^{2k}, \quad 1 \leq k \leq 4,$$

form an algebraically independent basis for the algebra  $H^*(\mathrm{BSO}(9); \mathbf{R}) =$  Weyl invariant polynomials on  $\mathfrak{t}$ : and then for any LICS given by  $\mathfrak{a} \subseteq \mathfrak{t} \otimes \mathbf{C}$  the bicomplex (4.1) is

$$(5.1) \quad \Lambda^*(\mathfrak{a}^*) \otimes \mathbf{C}[x_1, \dots, x_4]/(f_1, \dots, f_4).$$

Consider the 2-plane  $\mathfrak{a} \subseteq \mathfrak{h}$  spanned by

$$e_1 + e_2 + i\sqrt{2}e_3, \quad e_1 - e_2 + i\sqrt{2}e_4.$$

This plane is totally isotropic for the form  $f_1$  and hence does not have any real points—so it does correspond to an LICS. The cohomology of (5.1) is now computed by hand. One first considers the complex

$$Y^{**} = \Lambda^*(\bar{\mathfrak{a}}^*) \otimes \mathbf{C}[x_1, \dots, x_4]/(f_1, f_2, f_4)$$

and by standard “Koszul-type” arguments one finds

$$H^{*,*}(Y) = \mathbf{C}[\mathfrak{a}^*]/(f_2|_{\mathfrak{a}}, f_4|_{\mathfrak{a}}) \otimes \Lambda(\theta), \quad \text{bideg } \theta = (1, 2).$$

Then one works out the long exact sequence for

$$0 \rightarrow Y \xrightarrow{f} Y \rightarrow B \rightarrow 0.$$

What makes it “large” is that the ideal in the symmetric algebra  $\mathbf{C}[\mathfrak{a}^*]$  spanned by the restrictions  $f_j|_{\mathfrak{a}}$ ,  $1 \leq j \leq 4$ , has *three* (essential) generators which do *not* form a regular sequence.

If we had chosen  $\mathfrak{a} \subseteq \mathfrak{h}$  to be one of the planes lying over the twelve lines in  $\mathbf{CP}_3 = \mathbf{P}(\mathfrak{h})$  which comprise the intersection of the quadric  $f_1 = 0$  and the Fermat sextic  $f_3 = 0$  then the corresponding ideal in  $\mathbf{C}[\mathfrak{a}^*]$  has of course two generators  $f_2|_{\mathfrak{a}}, f_4|_{\mathfrak{a}}$  which form a regular pair: the cohomology of (5.1) is then appropriately small, and one does have  $E_2 = E_\infty$  in HdR for these twelve cases. I do not know of a general proof relating the syzygies of the Weyl invariant polynomials restricted to  $\mathfrak{a} \subseteq \mathfrak{h}$  and the nondegeneration of HdR.

6. The bicomplex (4.1) becomes quite unmanageable for explicit computations as the rank of  $G$  gets large. In §4 of our paper [3] we have offered another more tractable bicomplex which at least *conjecturally* computes (4.2). The construction of this conjectural bicomplex is motivated by the fact that the corresponding assertion for de Rham cohomology is true, *and* that for an admittedly special class of LICS on all classical families of even rank of type B, C, D as well as *all* LICS on groups of rank 2, and all the cases discussed above for  $\mathrm{SO}(9)$ , the conjecture is valid. We now describe this conjectural complex.

On any semisimple  $\mathfrak{g}$  of even rank fix an algebraically independent basis  $f_1, \dots, f_l$  for the Weyl invariants in the symmetric algebra  $\mathbf{R}[t^*]$ : let  $m_j = \deg f_j$ . Define

$$\Lambda^{*,*}(\varphi_1, \dots, \varphi_l) \otimes \mathbf{C}[\mathfrak{a}^*], \quad d\varphi_j = f_j|_{\mathfrak{a}},$$

where  $\mathrm{bideg} \varphi_j = (m_j, m_j - 1)$  and  $\mathfrak{a}^* \subseteq \mathbf{C}[\mathfrak{a}^*]$  has bidegree  $(1, 1)$ , so that  $\mathrm{bideg} f_j = (m_j, m_j)$  and  $\mathrm{bideg} d = (0, 1)$ . Then the conjecture is that the cohomology of this complex is (4.2). In any case the bicomplex above, as well as (4.1), depend only on the given  $\mathfrak{a} \subseteq \mathfrak{h} = \mathfrak{t} \otimes \mathbf{C}$  and the action of the Weyl group on  $\mathfrak{t}$ .

ADDED IN PROOF. After the submission of this manuscript I learned that counterexamples to the same conjecture of a quite different kind have been announced by L. Cordero, M. Fernandez and A. Gray in C.R. Acad. Sci. Paris Sér I **305** (1987), 753–756.

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DEPARTMENT OF MATHEMATICS, DUKE UNIVERSITY, DURHAM, NORTH CAROLINA 27706